Geometric & Quant Meths in Gravity & Particle Physics

Subtitle: **Off-diagonal deformations of Kerr black holes in Einstein and modified massive gravity**

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Brief summary of scientific activity; Recent publications; CERN visiting research & Master Class Program and Project IDEI

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page 2: Abstract

1. **Activity**: Project IDEI and visiting research program at CERN
Olivia Vacaru participation in the International Master class program

   high school students *' to get out and visit research institutes and perform volunteer work'*, http://www.physicsmasterclasses.org/.

2. **S. Vacaru scientific** and (pluralistic) pedagogical activity - 37 years - *Geometric and Quantum Methods in Gravity and Particle Physics*.

3. **Recent research**: general parameterizations for metrics and matter sources in GR and modified gravity with decoupling of field equations and exact generic off-diagonal solutions depending on all spacetime coordinates via classes of generating and integration functions

   generic off-diagonal nonlinear parametric interactions in GR mimicking effects in massive and/or modified gravity,

   distinguishing "generic" modified gravity solutions not encoded in GR.

page 3: Outline

1. Research Activity & Visibility
   - Summary of scientific and pedagogical activity
   - Important results
   - Comments on strategic and main directions
2. Decoupling & Integration of (Modified) Einstein eqs
   - Nonholonomic $f(R, T)$ gravity
   - d–metrics & d–connections
   - Effective/ modified Einstein eqs
   - Decoupling & off–diagonal integrals
   - Properties; LC–conditions; Non–Killing
3. Nonholonomic Deformations & Modified Kerr Metrics
   - Modifications/ deformations of Kerr metrics
   - Ellipsoidal and ellipsoid – de Sitter configurations

page 4: Summary of research and taching

Beginning of research activity in 1977 'geometric models of nuclear interactions', student at Tomsk Polytechnic University and young researcher at JINR, Dubna

Beginning of PhD on 'gauge gravity and twistors' at M. Lomonosov State University, Moscow, in 1984, research activity at Academy of Sciences of R. Moldova, Chişinău, and **defended thesis at UAIC Iaşi, 1994**

research and pluralistic university teaching on mathematical physics, geometric methods in particle physics and gravity, modifications and applications

Three "Strategic" Directions

1. **nonholonomic geometric flows evolutions and exact solutions** for Ricci solitons and field equations in (modified) gravity theories and cosmology
2. **geometric methods in quantization** of models with nonlinear dynamics and anisotropic field interactions
3. **(non) commutative geometry, almost Kähler and Clifford structures,** Dirac operators and effective Lagrange–Hamilton spaces and gravity

Inter-/ multi- disciplinary character of research: mathematics, physics, geometric methods, PDE and physics; stochastics and kinetics on curved spaces, applications ... Comments on 16 main directions of activity will follow.

page 5: Important results

Key points

- by 20 high level international and national programs, NATO and UNESCO;

  visiting/sabatical/associate professor fellowships and grants:

  CERN, USA, Germany, UK, Canada, Spain, Portugal, Romania etc.
- more than 130 short visits with lectures/ talks seminars (support from organizers)
- $>$ 140 scientific works (published and preprints, inspirehep.net)

  $>$ 70 high influence score (ISI Web Knowledge);

  individually - 50 % , with seniors - 30 % and young researchers - 20 %
- UAIC beginning June 2009: by 40 articles top ISI and high influent score

  "(red, yellow, blue)" competition of articles > 15 (5, 7, 3)

  $\sim$ 40 International Conferences/Seminars - host support from UK, Italy, Germany, France, Switzerland, Sweden, Spain, Belgium, Norway, Turkey...
- Hirsh factor - 16, more than 140 citations without self–citation
- grant IDEI 2011-15 "nonlinear dynamics and gravity"; visiting researcher CERN

page 6: 16 Main Research Directions

1. (Non) commutative gauge theories of gravity and generalizations and quantization

- (a) Affine and de Sitter models of gauge gravity.
- (b) Gauge like models of Einstein and Lagrange–Finsler gravity.
- (c) Locally anisotropic gauge theories and perturbative quantization.
- (d) Noncommutative gauge gravity.

2. Clifford structures and spinors on nonholonomic manifolds and bundles

- (a) Definition of spinors and Dirac operators on generalized Lagrange spaces.
- (b) Clifford structures with nonlinear connections and nonholonomic manifolds.
- (c) Nonholonomic Einstein–Dirac systems and extra dim gravity.
- (d) Nonholonomic gerbes, index theorems, and Clifford–Finsler algebroids.

works in R. Moldova (1994-95), and in JMP (1996), JHEP (1998); 3 monographs; a NATO workshop in Kiev (2001); 2p PLB (2001); JMP (2005); Collabor. with H. Dehnen (Germany) - DAAD, 2p in GRG (2003); 2p on two-connection perturb quant of gauge gravity (IJGMMP, 2010); collaborations with P. Stavrinos. G. Tsagas, Nadejda Vicol, F. C.

page 7: 16 Main Research Directions

3. Nearly autoparallel maps, twistors and conservation laws in nonholonomic pseudo–Riemannian, Lagrange and Finsler spaces

Development of directions in PhD thesis, together with S. Ostaf and I. Gottlieb and H. Dehnen, DAAD (1999-2000) and 2 GRG-2003. Supersymmetric generalizations in Monograph (Hadronic Press, 1998)

4. Locally anisotropic gravity in low energy limits of string/ brane theories; geometry of nonholonomic supermanifolds and super–Finsler space

- (a) Background methods and locally anisotropic (super) string/gravity
- (b) Supersymmetric generalizations of Lagrange–Finsler spaces.

Low energy limits of (super) strings to Lagrange-Finsler (AP NY; NP B, 1997) "Super-Finsler" term in Supersymmetry Encyclopedia.

5. Anisotropic Taub–NUT spaces and Dirac spin waves and solitonic solutions

Applications of the anholonomic deformation method for exact solutions in (with F. C. Popa and O. Ţînţăreanu; CQG, NPB, 2002) and Ricci flows (with M. Vişinescu, 2006)

page 8: 16 Main Research Directions

6. Anisotropic diffusion, kinetic/ thermodynamical processes, gravity/ mechanics

- (a) Stochastic processes, diffusion and thermodynamics on nonholonomic curved spaces (super) bundles.
- (b) Locally anisotropic kinetic processes and thermodynamics in curved spaces.

Îto and Stratonovich types of ansitoropic calculus: Annals of Physics (Leipzig, 2001); Annals of Physics (NY, 2001); supersymmetric generalization; Laplace operator and Ricci flows: Perelman's entropy and thermodynamical functions for Finsler-Ricci flows and evolutions (JMP, Rep. MP, IJGMMP, EPJH, 1996-2012).

7. Differential fractional derivative geometry, gravity and geometric mechanics, and deformation quantization

Collaboration with D. Baleanu (2010-2011): fractional derivative Einstein eqs and Ricci flows; 8 top ISI papers (J. Math. Phys, IJTP, Nonlin. Dyn., CEJP; Chaos, Solitons & Fractals) and 2 conferences in Turkey (Springer Proceedings, 2012).

8. Warped off–diagonal wormhole configurations, flux tubes and propagation of black holes in extra–dimensions

Geometric methods: solitonic and pp–wave solutions on off–diagonal generalization - collaboration with D. Singleton (California) and students, R. Moldova. 2 months in USA. Papers in PLB, 2 in CQG, JMP - 2002. Parts I, II (Balkan Geom. Press, 2005)

page 9: 16 Main Research Directions

9. Geometric methods of constructing generic off-diagonal solutions for Ricci solitons, nonholonomic Einstein spaces and in modified theories of gravity

- (a) Decoupling property of (generalized) Einstein equations and integrability for (modified) theories with commutative and noncommutative variables.
- (b) Generating exact solutions with ellipsoidal, solitonic and pp–wave configurations, possible cosmological solutions.
- (c) Generic off–diagonal Einstein–Yang–Mills–Higgs configurations.

Some tenths of papers in NPB, CQG, JMP, GRG, IJGMMP, IJMPA, IJMPD, JHEP, EJPC, IJTP etc

10. Solitonic gravitational hierarchies in Einstein and Finsler gravity

Collaboration with S. Anco, Canada; visiting international prof. in 2005-2006.
Encoding solutions of Einstein, Ricci flow eqs (and generalizations: Finsler etc) as bi-Hamilton structures and solitonic hierarchies.
Publications (Geom. & Phys., 2009) and (Acta Applicand. Math, 2010). Examples in CQG, JMP, IJTP, IJMMP, IJMPA and Parts I, II in (Balkan Geometry Press, 2005).

page 10: 16 Main Research Directions

11. Principles of Einstein–Finsler gravity and applications

- Classification of Lagrange–Finsler-affine spaces.
- Critical remarks and axiomatics of Einstein–Finsler gravity.
- Exact solutions in (non) commutative Finsler gravity and applications.
- (Non) commutative Finsler black branes, rings, ellipsoids, cosmological sols.

Generalized Einstein-Finsler eqs? Horvath (1950); Metric (non)compatible theories (ERE2010). Critical remarks (PLB 2010). Axiomatic EPS for EFG. Exact solutions for (non)commutative Finsler (CQG -2010,2011)

12. Stability of nonholonomic gravity and geometric flows with nonsymmetric metrics...

Nonsymmetric Ricci tensors $\rightarrow$ nonsymmetric metrics via nonholonomic Ricci flows.
A. Einstein (1925-1945) and L. P. Eisenhardt (1951-1952); G. Atanasiu and R. Miron for Finsler generalizations. J. W. Moffat (1984-95); Critics (S. Deser etc... 1993; T. Prokopec, 2006); My contributions (IJTP, SIGMA, 2008-2009), metric compatible constructions, proof: stability, Ricci flows

13. Covariant renormalizable anisotropic theories and exact solutions in gravity

- Modified dispersions, generalized Finsler structures and Hořava–Lifshitz theories on tangent bundles.
- Covariant renormalizable models for generic off–diagonal spacetimes and anisotropically modified gravity.

page 11: 16 Main Research Directions

14. Nonholonomic Ricci flows, thermodynamics & geometric mechanics; gravity and noncommutative geometry

1. Generalization of Perelman's functionals and Hamilton's equations for nonholonomic Ricci flows.
2. Statistical and thermodynamics for evolution of Lagrange–Finsler geometries and analogous gravity.
3. Nonholonomic Ricci solitons, exact solutions in gravity, and symmetric and nonsymmetric metrics; Geometric evolution of pp–wave and Taub NUT spaces.
4. Nonholonomic Dirac operators, distinguished spectral triples and evolution of models of noncommutative geometry and gravity theories.
5. Ricci solitons, modified gravity and quantization

A series of more than 10 papers (2005-2012) in JMP, IJGMMP, IJTP, Rep. MP, IJMPA ... Recent interest related to noncommutative/ modified gravity etc.

15. Geometric, Deformation, A-brane and two-connection gauge like quantization with almost Kähler models of (modified) gravity

- Almost Kähler and Lagrange–Finsler variables in geom mechanics and gravity.
- Geometric and DQ of Ricci flows, modified gravity and generalized Lagrange–Finsler and Hamilton–Cartan theories.
- Geometric and Fedosov quantization of Einstein gravity and modifications.
- A–brane quantization of gravity; two–connection quantization of Einstein, loops, and gauge gravity theories.

page 12: 16 Main Research Directions

16. Modified Gravity and Modern Cosmology

Most recent "fashion" and activity in inspirebeta.net

1. Geometric methods of constructing off–diagonal solutions in $f(R,T)$, bi–metric and massive gravity
2. Physics of generalized black holes, wormholes, rings and solitons
3. Classical and Quantum Gravity models on Tangent Lorentz Bundles
4. Cosmological solutions in bi–connection and bi–metric gravity theories
5. Off–diagonal ekypyrotic scenarios and equivalence of modified, massive and/or Einstein gravity
6. Modified dynamical supergravity breaking and off–diagonal super–Higgs effects
7. Ricci solitons, modified gravity and quantization

Recent publications in EJPC, J. Math. Phys. IJGMMP and a series of preprints etc.

page 13: Modified and Einstein Gravity and Ricci Solitons

Modifications of GR: $\nabla[\mathbf{g}] \to \mathbf{D}[\mathbf{g}]$; Lagrange density $R \to f(R,T)$

Vacuum MG: $f_R \mathbf{R}_{\alpha\beta} - \frac{1}{2} f \mathbf{g}_{\alpha\beta} + \left(\mathbf{g}_{\alpha\beta}\mathbf{D}_\gamma \mathbf{D}^\gamma - \mathbf{D}_\alpha \mathbf{D}_\beta\right) f_R = 0,$

for $f_R = \partial f/\partial R$. If $\mathbf{D} = \nabla$, vacuum $f(R)$ gravity.

generalized Ricci solitons: $\mathbf{R}_{\alpha\beta} + \mathbf{D}_\alpha \mathbf{D}_\beta K = \lambda \mathbf{g}_{\alpha\beta},$

$K = f_R$ and $\mathbf{D} \to \nabla$ and $\mathbf{g} \to \widehat{\mathbf{g}}$; stationary geometric flows; generalized Einstein spaces; bridge to QG.

MG with effective Newton/ cosmological "constants" & field eqs

$$\mathbf{R}_{\alpha\beta} = \Lambda(x^i, y^a)\mathbf{g}_{\alpha\beta},$$

"Polarized" cosmological constant $\Lambda = \frac{\lambda + \mathbf{D}_\gamma \mathbf{D}^\gamma f_R - f/2}{1 - f_R}.$

Generic off–diagonal solutions generated in explicit form for Killing symmetry, on $\partial/\partial y^4$ (for simplicity), when $\Lambda \approx \Lambda(x^i)$. Similarly, in can be included in Λ massive gravity effects.

page 14: Nonholonomic 2+2 splitting, and (n+n), or 2(3)+2+2+...

Aims: Find $e_\alpha = e_\alpha^{\alpha'} \partial_{\alpha'}$ when Einstein eqs for $(\mathbf{g}, \Gamma)$ decouple and can be integrated in very general forms:

Non–integrable (nonholonomic) 2+2 spacetime splitting in GR $(V, \mathbf{g})$,

4–d pseudo–Riemannian V, $\mathbf{g} = g_{\alpha\beta}$ with conventional $2+2$ splitting: indices $\alpha, \beta, ... = (i,a), (j,b), ...$ for $i,j,k,... = 1,2$; $a,b,c,.. = 3,4$; coordinates $u^\alpha = (x^i, y^a) = (x^1, x^2, y^3, y^4)$, or $u = (x,y)$, partial derivatives $\partial_\alpha := \partial/\partial u^\alpha$; $\partial_\alpha = (\partial_i, \partial_a)$

N–adapted frames/ bases: $\mathbf{N} : TV = hTV \oplus vTV$; $\mathbf{N} = N_i^a(x,y)\partial_a \otimes dx^i$

nonholonomic frames: $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = \mathbf{w}^\gamma{}_{\alpha\beta}(u)\, \mathbf{e}_\gamma$,

$$\mathbf{e}_\alpha := (\mathbf{e}_i = \partial_i - N_i^a \partial_a, e_b = \partial_b)$$
$$\mathbf{e}^\beta := (e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i)$$

anholonomy coefficients $\mathbf{w}^\gamma{}_{\alpha\beta}(u)$ are functionals of $N_i^a(x,y)$ and part deriv

page 15: N–adapted metrics

Frame transforms $\forall \mathbf{g}$ can be represented in equivalent forms:

1) With respect to coordinate bases: $\mathbf{g} = \underline{g}_{\alpha\beta}(u)\, du^{\alpha} \otimes du^{\beta}$

$$\text{for } \underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e \, g_{be} & g_{ab} \end{array} \right], \text{where } N_i^a \neq A_{bi}^a(x) y^b;$$

2) N–adapted, $\mathbf{g} = \mathbf{g}_{\alpha\beta}(u)\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta} = g_{ij}(x,y)\, e^i \otimes e^j + g_{ab}(x,y)\mathbf{e}^a \otimes \mathbf{e}^b$

In the simplest form, the decoupling property for metrics, $\mathbf{g}_{\alpha'\beta'}$, when ${}^K\mathbf{g}_{\alpha\beta} = e_{\alpha}^{\alpha'} e_{\beta}^{\beta'} \mathbf{g}_{\alpha'\beta'}$, $u^{\alpha} = (x^k, v, y^4)$, ansatz with Killing symmetry $\partial/\partial y^4$.

for $g_{\alpha\beta} = diag[g_i(x^k), h_a(x^k,v)]$ and $N_i^3 = w_i(x^k,v)$, $N_i^4 = n_i(x^k,v)$, $y^3 := v$

$$\begin{aligned} {}^K\mathbf{g} &= g_i(x^k) dx^i \otimes dx^i + h_a(x^k,v)\mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + w_i(x^k,v)dx^i, \qquad \mathbf{e}^4 = dy^4 + n_i(x^k,v)dx^i. \end{aligned}$$

page 16: Off–diagonal and N–adapted parameterizations of metrics

in coordinate frames

"∀" solution of generalized Einstein eqs, $g_{\alpha'\beta'}$, $\mathbf{g}_{\alpha\beta} = e^{\alpha'}_{\ \alpha} e^{\beta'}_{\ \beta} g_{\alpha'\beta'}, \rightarrow$

$$\mathbf{g}_{\alpha\beta} = \left| \begin{array}{cccc} g_1 + \omega^2({w_1}^2 h_3 + {n_1}^2 h_4) & \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & \omega^2\ w_1 h_3 & \omega^2\ n_1 h_4 \\ \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & g_2 + \omega^2({w_2}^2 h_3 + n_2^2 h_4) & \omega^2\ w_2 h_3 & \omega^2\ n_2 h_4 \\ \omega^2\ w_1 h_3 & \omega^2\ w_2 h_3 & \omega^2 h_3 & 0 \\ \omega^2\ n_1 h_4 & \omega^2\ n_2 h_4 & 0 & \omega^2 h_4 \end{array} \right|$$

$$\begin{array}{rcl} \textbf{N–adapted } \mathbf{g} & = & g_i dx^i \otimes dx^i + \omega^2 h_a \underline{h}_a \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 & = & dy^3 + (w_i + \underline{w}_i)\, dx^i,\ \mathbf{e}^4 = dy^4 + (n_i + \underline{n}_i)\, dx^i, \end{array}$$

$g_i = g_i(x^k), g_a = \omega^2(x^i, y^c)\ h_a(x^k, y^3)\underline{h}_a(x^k, y^4)$, not summation on "a",
$N_i^3 = w_i(x^k, y^3) + \underline{w}_i(x^k, y^4), N_i^4 = n_i(x^k, y^3) + \underline{n}_i(x^k, y^4)$,
are functions of necessary smooth class generating solutions of gravitat. field eqs.

page 17: Connections and (generalized) Einstein eqs in N–adapted form

$$\mathbf{g} \to \begin{array}{lll} \nabla : & \nabla \mathbf{g} = 0;\ {}^{\nabla}\mathcal{T}^{\alpha} = 0, & \text{the Levi–Civita connection ;} \\ \widehat{\mathbf{D}} : & \widehat{\mathbf{D}}\mathbf{g} = 0;\ h\widehat{\mathcal{T}}^{\alpha} = 0, v\widehat{\mathcal{T}}^{\alpha} = 0, & \text{the canonical d–connection} \end{array}$$

$$\widehat{\mathbf{D}}[\mathbf{g}] = \nabla[\mathbf{g}] + \widehat{\mathbf{Z}}[\mathbf{g}]$$

"auxiliary" connection $\widehat{\mathbf{D}} = \widehat{\Gamma}^{\gamma}_{\ \alpha\beta} = (\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc})$: 1) $\widehat{\mathbf{D}}\mathbf{g} = 0$, 2) $\widehat{T}^{i}_{\ jk} = 0, \widehat{T}^{a}_{\ bc} = 0$.

Torsion $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$: $\widehat{T}^{i}_{\ ja} = \widehat{C}^{i}_{jb}, \widehat{T}^{a}_{\ ji} = -\Omega^{a}_{\ ji}, \widehat{T}^{c}_{\ aj} = \widehat{L}^{c}_{aj} - e_a(N^c_j)$.

$$\widehat{L}^{i}_{jk} = \frac{1}{2}g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),\ \widehat{L}^{a}_{bk} = e_b(N^a_k) + \frac{1}{2}g^{ac}(e_k g_{bc} - g_{dc}\, e_b N^d_k - g_{db}\, e_c N^d_k),$$
$$\widehat{C}^{i}_{jc} = \frac{1}{2}g^{ik} e_c g_{jk},\ \widehat{C}^{a}_{bc} = \frac{1}{2}g^{ad}(e_c g_{bd} + e_c g_{cd} - e_d g_{bc})$$

N–adapted Einstein eqs: $\widehat{\mathbf{R}}_{\ \beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\ {}^{s}R = \Upsilon_{\beta\delta},$

LC–conditions for GR: $\widehat{L}^{c}_{aj} = e_a(N^c_j),\ \widehat{C}^{i}_{jb} = 0,\ \Omega^{a}_{\ ji} = 0,$

$\widehat{\mathbf{R}}_{\ \beta\delta}$ for $\widehat{\Gamma}^{\gamma}_{\ \alpha\beta}$, ${}^{s}R = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\ \beta\delta}$ and $\Upsilon_{\beta\delta} \to \varkappa T_{\beta\delta}$ for $\widehat{\mathbf{D}} \to \nabla$.

page 18: Decoupling in MG and GR

Theorem 1 (Decoupling): effective Einstein eqs for ${}^{K}\mathbf{g}$ and $\Lambda(x^i,\theta)$,

with $a^\bullet = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial v$, parameters θ, for $h^*_{3,4} \neq 0, \Lambda \neq 0$, $g_i = \varepsilon_i e^{\psi(x^i)}$, are

$$\begin{aligned}
\varepsilon_1 \ddot{\psi} + \varepsilon_2 \psi'' &= 2\Lambda \\
\phi^*\, h_4^* &= 2 h_3 h_4 \Lambda \\
\beta w_i + \alpha_i &= 0 \\
n_i^{**} + \gamma n_i^* &= 0
\end{aligned}$$

for $\alpha_i = h_4^* \partial_i \phi, \beta = h_4^*\ \phi^*, \gamma = \left(\ln \frac{|h_4|^{3/2}}{|h_3|} \right)^*$

generating function $\phi = \ln \left| \frac{h_4^*}{\sqrt{|h_3 h_4|}} \right|$

Remarks: 1) do not "see" decoupling for the LC in non–N–adapted frames.

2) $\exists$ decoupling for non–Killing ansatz and $h_3^* = 0$, or $h_4^* = 0$

page 19: Constructing off–diagonal "one–Killing" solutions

Theorem 2 (Integral Varieties)

$$g_i = \varepsilon_i e^{\psi},$$

$$h_3 = {}^0h_3 \left[1 + (e^{\phi})^*/2\Lambda\sqrt{|{}^0h_3|}\right]^2, \ h_4 = {}^\circ h_4 \exp[e^{2\ \phi}/8\Lambda]$$

$$w_i = \partial_i \phi/\phi^*$$

$$n_k = {}_1n_k + {}_2n_k \int [h_3/(\sqrt{|h_4|})^3] dv$$

generating functions $\psi(x^k,\theta), \phi(x^k,v,\theta)$; source $\Lambda(x^k,\theta)$,
integration functions ${}^0h_a(x^k,\theta),\ {}_1n_k(x^k,\theta),\ {}_2n_k(x^k,\theta)$

"slight violation" of decoupling for the LC conditions

$$w_i^* = \mathbf{e}_i \ln|h_4|, \mathbf{e}_k w_i = \mathbf{e}_i w_k, \qquad n_i^* = 0,\ \partial_i n_k = \partial_k n_i \to {}_2n_i = 0.$$

page 20: Nonholonomic deformations 'prime' $\rightarrow$ 'target'

Dependence on y^4, "vertical" conformal $\omega^2(x^j, v, y^4), \partial a/\partial y^4 := a^\circ$,

$\omega^2 = 1$ results in solutions with Killing symmetry,

$$\mathbf{g} = g_i(x^k)dx^i \otimes dx^i + \omega^2(x^j, v, y^4)h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a,$$
$$\mathbf{e}^3 = dy^3 + w_i(x^k, v)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k, v)dx^i,$$
$$\mathbf{e}_k\omega = \partial_k\omega + w_k\omega^* + n_k\omega^\circ = 0.$$

N–deformations & gravitational polarizations $\eta_\alpha, \ \eta_i^a$,

N–deforms, ${}^\star\mathbf{g} = [\,{}^\star g_i, \ {}^\star h_a, \ {}^\star N_k^a] \rightarrow \ {}^\eta\mathbf{g} = [\, g_i, h_a, N_k^a]$,

$${}^\eta\mathbf{g} = \eta_i(x^k, v)\ {}^\star g_i(x^k, v)dx^i \otimes dx^i + \eta_a(x^k, v)\ {}^\star h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a,$$

$$\mathbf{e}^3 = dv + \eta_i^3(x^k, v)\ {}^\star w_i(x^k, v)dx^i, \ \mathbf{e}^4 = dy^4 + \eta_i^4(x^k, v)\ {}^\star n_i(x^k, v)dx^i.$$

For a solution in GR with well–defined boundary/ asymptotic conditions, we can search ${}^\star\mathbf{g} \rightarrow \ {}^\eta\mathbf{g}$ to a "parametric/noncommutative/stochastic ..." solution in GR, MG.

page 21: Nonholonomic Deformations & Modified Kerr Metrics

Generating the Kerr vacuum solution: Boyer–Linquist coordinates $(r, \vartheta, \varphi, t)$,

for $r = m_0(1 + p\widehat{x}_1)$, $\widehat{x}_2 = \cos\vartheta$; parameters p, q; total black hole mass, m_0 (not confused with μ_g in massive gravity); total angular momentum, am_0, for the asymptotically flat, stationary and axisymmetric Kerr spacetime. $m_0 = Mp^{-1}$ and $a = Mqp^{-1}$, $p^2 + q^2 = 1$ implies $m_0^2 - a^2 = M^2$, the vacuum solution

$$
\begin{aligned}
ds^2_{[0]} &= (dx^{1'})^2 + (dx^{2'})^2 + \overline{A}(\mathbf{e}^{3'})^2 + (\overline{C} - \overline{B}^2/\overline{A})(\mathbf{e}^{4'})^2, \\
\mathbf{e}^{3'} &= dt + d\varphi \overline{B}/\overline{A} = dy^{3'} - \partial_{i'}(\widehat{y}^{3'} + \varphi \overline{B}/\overline{A})dx^{i'}, \mathbf{e}^{4'} = dy^{4'} = d\varphi,
\end{aligned}
$$

for $x^{1'}(r,\vartheta),\ x^{2'}(r,\vartheta),\ y^{3'} = t + \widehat{y}^{3'}(r,\vartheta,\varphi) + \varphi\overline{B}/\overline{A}, y^{4'} = \varphi,\ \partial_\varphi \widehat{y}^{3'} = -\overline{B}/\overline{A}$,
for which $(dx^{1'})^2 + (dx^{2'})^2 = \Xi\left(\Delta^{-1}dr^2 + d\vartheta^2\right)$,

$$
\begin{aligned}
\overline{A} &= -\Xi^{-1}(\Delta - a^2\sin^2\vartheta), \overline{B} = \Xi^{-1} a \sin^2\vartheta\left[\Delta - (r^2 + a^2)\right], \\
\overline{C} &= \Xi^{-1}\sin^2\vartheta\left[(r^2 + a^2)^2 - \Delta a^2 \sin^2\vartheta\right],\ \Delta = r^2 - 2m_0 + a^2,\ \Xi = r^2 + a^2\cos^2\vartheta. \\
\text{Prime data }\ \mathring{g}_{1'} &= \mathring{g}_{2'} = 1, \mathring{h}_{3'} = \overline{A}, \mathring{h}_{4'} = \overline{C} - \overline{B}^2/\overline{A},\ \mathring{N}^3_{i'} = \mathring{n}_{i'} = -\partial_{i'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A}), \mathring{N}^4_{i'} = \mathring{w}_{i'} = 0
\end{aligned}
$$

Kerr vacuum solution is a "degenerate" case of 4–d off–diagonal vacuum solutions; primary metrics with diagonal coefficients depending only on two "horizontal" N–adapted coordinates; off–diagonal terms induced by rotation.

page 22: Nonholonomic Deformations & Modified Kerr Metrics

Deformations of Kerr metrics in 4–d massive gravity

Goal: construct $(\mathbf{g}, \mathbf{N}, {}^{v}\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \rightarrow (\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, {}^{v}\widetilde{\Upsilon} = \widetilde{\lambda}, \widetilde{\Upsilon} = \widetilde{\lambda}), \widetilde{\lambda} = const \neq 0$.
Condition: target metric **g** positively defines a generic off–diagonal solutions in 4–d massive gravity.

$$ds^2 = e^{\psi(x^{k'})}[(dx^{1'})^2 + (dx^{2'})^2] - \frac{e^{2\varpi}}{4\mu_g^2\,|\widetilde{\lambda}|}\overline{A}[dy^{3'} + \left(\partial_{k'}\ {}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right) dx^{k'}]^2$$
$$+ \frac{(\varpi^*)^2}{\mu_g^2\ \lambda(x^{k'})}(\overline{C} - \overline{B}^2/\overline{A})[d\varphi + (\partial_{i'}\ {}^{\eta}\widetilde{A})dx^{i'}]^2.$$

The gravitational polarizations (η_i, η_a), $e^{\psi(x^k)} = \widetilde{\eta}_{1'} = \widetilde{\eta}_{2'}$, $\widetilde{\eta}_{3'} = \frac{e^{2\varpi}}{4\mu_g^2\,|\widetilde{\lambda}|}$, $\widetilde{\eta}_{4'} = \frac{(\varpi^*)^2}{\mu_g^2\ \lambda(x^{k'})}$,
and N–coefficients are $w_{i'} = \mathring{w}_{i'} + {}^{\eta}w_{i'} = \partial_{i'}({}^{\eta}\widetilde{A}[\varpi])$, $n_{k'} = \mathring{h}_{k'} + {}^{\eta}n_{k'} = \partial_{k'}(-\widehat{y}^{3'} + \varphi\overline{B}/\overline{A} + {}^{\eta}n)$,
where ${}^{\eta}\widetilde{A}(x^k, y^4)$ is introduced to satisfy LC–conditions and $\psi^{\bullet\bullet} + \psi'' = 2\,\mu_g^2\,\lambda(x^{k'})$.
For N–coefficients, $\check{\Phi} = \exp[\varpi(x^{k'}, y^4)]\sqrt{|\mathring{h}_{3'}|}$, when $\mathring{h}_{3'}\mathring{h}_{4'} = \overline{A}\overline{C} - \overline{B}^2$ and
$w_{i'} = \mathring{w}_{i'} + {}^{\eta}w_{i'} = \partial_{i'}(\,e^{\varpi}\sqrt{|\overline{A}\overline{C} - \overline{B}^2|})/\,\varpi^* e^{\varpi}\sqrt{|\overline{A}\overline{C} - \overline{B}^2|} = \partial_{i'}\ {}^{\eta}\widetilde{A}$.
The solutions are for stationary LC–configurations determined by off–diagonal massive gravity effects on Kerr black holes; new class of spacetimes are with Killing symmetry on $\partial/\partial y^{3'}$ and generic dependence on $(x^{i'}(r,\vartheta), \varphi)$.

page 23: Nonholonomic Deformations & Modified Kerr Metrics

Small f–modifications of Kerr metrics and massive gravity

A "prime" solution for massive gravity/ effective modelled in GR with source ${}^{\mu}\Lambda = \mu_g^2\, \lambda(x^{k'})$, or re–defined to ${}^{\mu}\tilde{\Lambda} = \mu_g^2\, \tilde{\lambda} = const$. Adding a "small" value $\widetilde{\Lambda}$ determined by f–modifications, we work in N–adapted frames with an effective source $\Upsilon = \widetilde{\Lambda} + \widetilde{\lambda}$. Construct off–diagonal solutions in modified f–gravity generated from the Kerr black hole solution as a result of two deformations
$(\mathbf{g}, \mathbf{N}, {}^{v}\mathring{\Upsilon} = 0, \mathring{\Upsilon} = 0) \rightarrow (\widetilde{\mathbf{g}}, \widetilde{\mathbf{N}}, {}^{v}\widetilde{\Upsilon} = \widetilde{\lambda}, \widetilde{\Upsilon} = \widetilde{\lambda}) \rightarrow ({}^{\varepsilon}\mathbf{g}, {}^{\varepsilon}\mathbf{N}, \check{} = \varepsilon\,\widetilde{\Lambda} + {}^{\mu}\tilde{\Lambda}, \Upsilon = \varepsilon\,\widetilde{\Lambda} + {}^{\mu}\tilde{\Lambda})$,
when the target data $\mathbf{g} = {}^{\varepsilon}\mathbf{g}$ and $\mathbf{N} = {}^{\varepsilon}\mathbf{N}$ depend on a small parameter ε, $0 < \varepsilon \ll 1$, $|\varepsilon\,\widetilde{\Lambda}| \ll |{}^{\mu}\tilde{\Lambda}|$, i.e. consider that f–modifications in N–adapted frames are much smaller than massive gravity effects (in a similar from, we can analyze nonlinear interactions with $|\varepsilon\,\widetilde{\Lambda}| \gg |{}^{\mu}\tilde{\Lambda}|$). N–adapted transforms $[\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i] \rightarrow$
$[g_i = (1+\varepsilon\chi_i)\widetilde{\eta}_i\mathring{g}_i, h_3 = (1+\varepsilon\chi_3)\widetilde{\eta}_3\mathring{h}_3, h_4 = (1+\varepsilon\chi_4)\widetilde{\eta}_4\mathring{h}_4, {}^{\varepsilon}w_i = \mathring{w}_i + \widetilde{w}_i + \varepsilon\overline{w}_i, {}^{\varepsilon}n_i = \mathring{n}_i + \tilde{n}_i + \varepsilon\overline{n}_i]$;
$\Upsilon = {}^{\mu}\tilde{\Lambda}(1 + \varepsilon\,\widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda})$; ${}^{\varepsilon}\tilde{\Phi} = \tilde{\Phi}(x^k,\varphi)[1 + \varepsilon\;{}^{1}\tilde{\Phi}(x^k,\varphi)/\tilde{\Phi}(x^k,\varphi)] = \exp[{}^{\varepsilon}\varpi(x^k,\varphi)]$.
A new class of ε–deformed solutions with $\chi_1 = \chi_2 = \chi$, for $\partial_{11}\chi + \epsilon_2\partial_{22}\chi = 2\widetilde{\Lambda}$; $\chi_3 = 2\,{}^{1}\tilde{\Phi}/\tilde{\Phi} - \widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda}$,
$\chi_4 = 2\partial_4\,{}^{1}\tilde{\Phi}/\tilde{\Phi} - 2\,{}^{1}\tilde{\Phi}/\tilde{\Phi} - \widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda}$, $\overline{w}_i = (\frac{\partial_i\,{}^{1}\tilde{\Phi}}{\partial_i\tilde{\Phi}} - \frac{\partial_4\,{}^{1}\tilde{\Phi}}{\partial_4\tilde{\Phi}})\frac{\partial_i\tilde{\Phi}}{\partial_4\tilde{\Phi}} = \partial_i\;{}^{1}\check{A}$, $\overline{n}_i = 0$, and $\mathring{h}_{3'}\mathring{h}_{4'} = \overline{AC} - \overline{B}^2$.
Two generating functions $\tilde{\Phi} = e^{\varpi}$ and ${}^{1}\tilde{\Phi}$ and two sources ${}^{\mu}\tilde{\Lambda}$ and $\widetilde{\Lambda}$. Putting together, off–diagonal generalization of the Kerr metric by "main" mass gravity terms and additional ε–parametric f–modifications,

$$ds^2 = e^{\psi(x^{k'})}(1+\varepsilon\chi(x^{k'}))[(dx^{1'})^2 + (dx^{2'})^2]$$
$$-\frac{e^{2\varpi}}{4|{}^{\mu}\tilde{\Lambda}|}\overline{A}[1+\varepsilon(2e^{-\varpi}\;{}^{1}\tilde{\Phi} - \widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda})][dy^{3'} + \left(\partial_{k'}\;{}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right)dx^{k'}]^2$$
$$+\frac{(\varpi^*)^2}{{}^{\mu}\tilde{\Lambda}}(\overline{C} - \overline{B}^2/\overline{A})[1+\varepsilon(2e^{-\varpi}\partial_4\,{}^{1}\tilde{\Phi} - 2e^{-\varpi}\;{}^{1}\tilde{\Phi} - \widetilde{\Lambda}/{}^{\mu}\tilde{\Lambda})][d\varphi + (\partial_{i'}\widetilde{A} + \varepsilon\partial_{i'}\;{}^{1}\check{A})dx^{i'}]^2.$$

page 24: Ellipsoidal 4–d deformations of the Kerr metric

Vacuum ellipsoidal configurations

A model when f–modifications compensate massive gravity deformations of a Kerr solution, with $\Upsilon = {}^{\mu}\tilde{\Lambda} + \varepsilon\widetilde{\Lambda} = 0$, and result in ellipsoidal off–diagonal configurations in GR, where $\varepsilon = -{}^{\mu}\tilde{\Lambda}/\widetilde{\Lambda} \ll 1$ can be considered as an eccentricity parameter.

$$ds^2 = e^{\psi(x^{k'})}(1+\varepsilon\chi(x^{k'}))[(dx^{1'})^2 + (dx^{2'})^2] - \frac{e^{2\varpi}}{4\mu_g^2|\tilde{\lambda}|}\overline{A}[1+\varepsilon\chi_{3'}][dy^{3'} + (\partial_{k'}\,{}^{\eta}n(x^{i'})$$
$$-\partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A}))dx^{k'}]^2 + \frac{(\partial_4\varpi)^2\eta_{4'}}{\mu_g^2\,\tilde{\lambda}}(\overline{C} - \overline{B}^2/\overline{A})[1+\varepsilon\chi_{4'}][d\varphi + (\partial_{i'}\widetilde{A} + \varepsilon\partial_{i'}\,{}^{1}\check{A})dx^{i'}]^2,$$

LC–conditions: $\mathbf{e}_i \ln\sqrt{|h_3|} = 0,\ \partial_i w_j = \partial_j w_i\ w_{i'} = \partial_{i'}\,{}^{\varepsilon}\Phi/\partial_{\varphi}\,{}^{\varepsilon}\Phi = \partial_{i'}(\widetilde{A} + \varepsilon\,{}^{1}\check{A}),\ {}^{\varepsilon}\Phi = \exp(\varpi + \varepsilon\chi_{3'})$.

Choose such $\chi_{3'}$, when $h_{3'} = 0$ defines a stationary rotoid configuration (different from to the ergo sphere for the Kerr solutions): Prescribing $\chi_{3'} = 2\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)$, for constant parameters $\underline{\zeta}$, ω_0 and φ_0, and introducing $\overline{A}(r,\vartheta)[1+\varepsilon\chi_{3'}(r,\vartheta,\varphi)] = \widehat{A}(r,\vartheta,\varphi) = -\Xi^{-1}(\widehat{\Delta} - a^2\sin^2\vartheta),\ \widehat{\Delta}(r,\varphi) = r^2 - 2m(\varphi) + a^2$, as ε–deformations of Kerr coefficients, we get an effective "anisotropically polarized" mass

$$m(\varphi) = m_0/\left(1 + \varepsilon\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)\right).$$

The condition results in an ellipsoidal "deformed horizon" $r(\vartheta,\varphi) = m(\varphi) + \left(m^2(\varphi) - a^2\sin^2\vartheta\right)^{1/2}$.

For $a = 0$, this is just the parametric formula for an ellipse with eccentricity ε, $r_+ = \frac{2m_0}{1+\varepsilon\underline{\zeta}\sin(\omega_0\varphi+\varphi_0)}$.

page 25: Ellipsoidal 4–d deformations of the Kerr metric

Ellipsoid Kerr – de Sitter configurations

A subclass of solutions with rotoid configurations if we constrain χ_3 from the ε–deformations in the form

$$\chi_3 = 2\, {}^1\tilde{\Phi}/\tilde{\Phi} - \widetilde{\Lambda}/\, {}^{\mu}\tilde{\Lambda} = 2\underline{\zeta}\sin(\omega_0\varphi + \varphi_0),$$

${}^1\tilde{\Phi} = e^{\varpi}[\widetilde{\Lambda}/2\, {}^{\mu}\tilde{\Lambda} + \underline{\zeta}\sin(\omega_0\varphi + \varphi_0)]$, for $\tilde{\Phi} = e^{\varpi}$, we generate ellipsoid Kerr – de Sitter configurations

$$\begin{aligned} ds^2 &= e^{\psi(x^{k'})}(1 + \varepsilon\chi(x^{k'}))[(dx^{1'})^2 + (dx^{2'})^2] - \\ &\frac{e^{2\varpi}}{4|\, {}^{\mu}\tilde{\Lambda}|}\overline{A}[1 + 2\varepsilon\underline{\zeta}\sin(\omega_0\varphi + \varphi_0)][dy^{3'} + \left(\partial_{k'}\, {}^{\eta}n(x^{i'}) - \partial_{k'}(\widehat{y}^{3'} + \varphi\overline{B}/\overline{A})\right) dx^{k'}]^2 + \\ &\frac{(\varpi^*)^2}{{}^{\mu}\tilde{\Lambda}}(\overline{C} - \overline{B}^2/\overline{A})[1 + \varepsilon(\partial_4\varpi\, \widetilde{\Lambda}/\widetilde{\lambda} + 2\partial_4\varpi\underline{\zeta}\sin(\omega_0\varphi + \varphi_0) + 2\omega_0\, \underline{\zeta}\cos(\omega_0\varphi + \varphi_0))] \\ &[d\varphi + (\partial_{i'}\widetilde{A} + \varepsilon\partial_{i'}\, {}^1\check{A})dx^{i'}]^2. \end{aligned}$$

Such metrics are with Killing symmetry on $\partial/\partial y^3$ and completely defined by a generating function $\varpi(x^{k'}, \varphi)$ and sources ${}^{\mu}\tilde{\Lambda} = \mu_g^2\,\lambda$ and $\widetilde{\Lambda}$. They define ε–deformations of Kerr – de Sitter black holes into ellipsoid configurations with effective (polarized) cosmological constants determined by constants in massive gravity and f–modifications.

page 26: **Conclusions**

- Almost 37 years research activity in theoretical and mathematical physics; geometric methods in physics and geometric mechanics; kinetics and geometric thermodynamics; applied mathematics, differential geometry, nonholonomic geometric analysis and evolution eqs.
- Recent associated activity at CERN, project IDEI, International Master class programs for 15-19 years students.
- Decoupling & Integration of Modified Einstein equations and Ricci solitons; applications in modern cosmology and astrophysics
- Mathematical physics aspects related to geometric, deformation, A–brane and gauge like quantization of gravity and nonholonomic geometric evolution theories.
- Future interest for supersymmetric models and noncommutative geometry.