Non–commutative Algebroid Ricci Flows, Modified Gravity & Deformation Quantization

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Motivation and Goals

Outline some directions of recent and future activity
Aim: establish collaborations on mathematical and theoretical physics, geometry and physics

Four Directions

1. **nonholonomic geometric flows evolution**: (non) commutative spaces, Lie algebroids with N–connections, almost symplectic structures
2. **(non) commutative geometry, almost Kähler and Clifford structures,** Dirac operators and effective Einstein and Lagrange–Finsler spaces
3. **Ricci solitons in non–Riemanian geometry, modified gravity** and PDEs decoupling and off–diagonal solutions
4. **geometric methods in quantization** of models with nonholonomic nonlinear dynamics and anisotropic field interactions

slide 4: **Almost Kähler Models for Einstein & Lagrange–Finsler Spaces**

Preliminaries: nonholonomic manifolds and bundles

Nonholonomic manifold: $(V, \mathcal{N})$, [dim $V = n + m$ with finite $n, m \geq 2$]

$$\mathbf{N}: \ TV = hTV \oplus vTV$$

$\mathbf{V} = (V, \mathbf{N}, \mathbf{g})$, **TV**, or $\mathbf{TM} = (TM, \mathbf{N}, L)$, example 2+2+2+.... splitting

N–connection: C. Ehresmann – 1955, E. Cartan – 1935

Lagrange space (TM, L), J. Kern - 1974

regular Lagrangian $\mathcal{L} = L(x,y)$, $\tilde{h}_{ab} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^a \partial y^b}$, det $|\tilde{h}_{ab}| \neq 0$

Finsler geometry: $L = F^2(x,y)$, $F(x, \xi y) = \xi F(x,y), \xi > 0$

Theorem: $\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial y^i} - \frac{\partial \mathcal{L}}{\partial x^i} = 0$ equivalent $\frac{dx^a}{d\tau} + 2\tilde{G}^a(x,y) = 0$

$y^i = dx^i/d\tau$, for $x^i(\tau)$, $S = y^i \frac{\partial \mathcal{L}}{\partial x^i} - 2\tilde{G}^a \frac{\partial}{\partial y^a}$

when $\tilde{G}^a = \frac{1}{4}\ \tilde{h}^{a\ n+i}(\frac{\partial^2 \mathcal{L}}{\partial y^{n+i}\partial x^k} y^{n+k} - \frac{\partial \mathcal{L}}{\partial x^i})$

N–coefficients: $\tilde{N}_i^a = \frac{\partial \tilde{G}^a}{\partial y^{n+i}}$, $\mathbf{N} = N_i^a(x,y)dx^i \otimes \partial/\partial y^a$

slide 5:

Sasaki lift and canonical d–connections

Corollary: Sasaki lift from M to TM :
prescribed $\mathcal{L} \rightarrow \tilde{\mathbf{g}} = h\tilde{\mathbf{g}} + v\tilde{\mathbf{g}}$, $(\mathbf{g}, \mathbf{N}) \sim (\tilde{\mathbf{g}}, \tilde{\mathbf{N}}[\mathcal{L}])$

Def. d–connection $\mathbf{D} = (hD; vD)$, metric compatible if $\mathbf{Dg} = 0$

Def. Ricci, $\mathbf{R}ic$, and Einstein, $\mathbf{E}$, of $\mathbf{D}$; d–tens, d–vect: $\mathbf{Y} = hY + vY$

Theor. $\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$ and $\tilde{\mathbf{D}} = \nabla + \tilde{\mathbf{Z}}$

$$\mathbf{g} = \tilde{\mathbf{g}} \rightarrow \begin{array}{lccc} \nabla : & \nabla \mathbf{g} = 0; \ {}^{\nabla}\mathcal{T}^{\alpha} = 0, & \text{Levi–Civita connection} \\ \widehat{\mathbf{D}} : & \widehat{\mathbf{D}}\mathbf{g} = 0; \ h\widehat{\mathcal{T}}^{\alpha} = 0, v\widehat{\mathcal{T}}^{\alpha} = 0, & \text{canonical d–connection} \\ \tilde{\mathbf{D}} : & \tilde{\mathbf{D}}\mathbf{g} = 0; \ h\tilde{\mathcal{T}}^{\alpha} = 0, v\tilde{\mathcal{T}}^{\alpha} = 0, & \text{Cartan d–connection} \end{array}$$

$\mathbf{g} = \{\mathbf{g}_{\alpha'\beta'}\} \rightarrow \tilde{\mathbf{g}}_{\alpha\beta} = [\, \tilde{g}_{ij}, \tilde{h}_{ab}], \tilde{\mathbf{e}}_{\alpha} = e^{\alpha'}_{\ \alpha} e_{\alpha'}$ and $\mathbf{g}_{\alpha'\beta'} e^{\alpha'}_{\ \alpha} e^{\beta'}_{\ \beta} = \tilde{\mathbf{g}}_{\alpha\beta}$
$\tilde{\mathbf{g}} = \tilde{g}_{ij} dx^i \otimes dx^j + \tilde{h}_{ab} \tilde{\mathbf{e}}^a \otimes \tilde{\mathbf{e}}^b, \ \tilde{g}_{ij} = \tilde{h}_{n+i\ n+j},$
$\tilde{\mathbf{e}}_{\alpha} = (\tilde{\mathbf{e}}_i = \partial_i - \tilde{N}^a_i \partial_a, e_a = \partial_a), \tilde{\mathbf{e}}^{\alpha} = (e^i = dx^i, \tilde{\mathbf{e}}^a = dy^a + \tilde{N}^a_i dx^i)$

slide 6: **Canonical almost Kähler variables, semi–Riemannian & LF**

Canon alm complex: $\tilde{\mathbf{J}}(\tilde{\mathbf{e}}_i) = -e_{2+i}$ & $\tilde{\mathbf{J}}(e_{2+i}) = \tilde{\mathbf{e}}_i$, $\tilde{\mathbf{J}} \circ \tilde{\mathbf{J}} = -\mathbb{I}$
Neijenhuis: $^{\mathbf{J}}\mathbf{\Omega}(\mathbf{X}, \mathbf{Y}) := -[\mathbf{X}, \mathbf{Y}] + [\mathbf{JX}, \mathbf{JY}] - \mathbf{J}[\mathbf{JX}, \mathbf{Y}] - \mathbf{J}[\mathbf{X}, \mathbf{JY}]$

Prop.-Def. $\forall\ (\mathbf{g}, \mathbf{J}) \rightarrow$ almost sympl struct $\theta(\mathbf{X}, \mathbf{Y}) := \mathbf{g}\,(\mathbf{JX}, \mathbf{Y})$

- almost Hermitian model of $(\mathbf{V}, \mathbf{g}, \mathbf{N})$:
  $\mathbf{H}^{n+n} = (\mathbf{V}, \theta(\cdot,\cdot) := \mathbf{g}\,(\mathbf{J}\cdot,\cdot)\,, \mathbf{J})$
- $\mathbf{H}^{n+n}$ is almost Kähler, $\mathbf{K}^{n+n}$, if and only if $d\theta = 0$

Theor.: The Cartan d–connection $\tilde{\mathbf{D}}$ is a unique almost symplectic d–connection: $\tilde{\mathbf{D}}\tilde{\theta} = 0$ and $\tilde{\mathbf{D}}\tilde{\mathbf{J}} = 0$

Theor.: Einstein manifold $Ric = \lambda \mathbf{g}$
canonical variables, $\widehat{\mathbf{R}}ic = \lambda \mathbf{g}$ and $\widehat{\mathbf{Z}} = 0$
Cartan type almost Kähler variables, $\tilde{\mathbf{R}}ic = \lambda \mathbf{g}$ and $\tilde{\mathbf{Z}} = 0$

slide 7: **almost Kähler Lie Algebroids and N–connections**

Def.: d–algebroid: $\mathcal{E} = (\mathbf{E}, \lfloor\cdot,\cdot\rfloor, \rho)$ over a manifold M :
1) $\mathbf{N}$: $TE = hE \oplus vE$
2) Lie aglebroid structure: 2a) a real *vector bundle* $\tau : \mathbf{E} \to M$;
2b) a *Lie bracket* $\lfloor\cdot,\cdot\rfloor$ on $Sec(\tau)$ of map τ
2c) *anchor* map $\rho : \mathbf{E} \to TM$, $\rho : Sec(\tau) \to \mathcal{X}(M)$ of $C^\infty(M)$–modules

$$\lfloor X, fY\rfloor = f\lfloor X, Y\rfloor + \rho(X)(f)Y, \ \forall X, Y \in Sec(\tau) \text{ and } f \in C^\infty(M)$$

ρ equivalent to a homomorphism between the Lie algebras $(Sec(\tau), \lfloor\cdot,\cdot\rfloor)$ and $(\mathcal{X}(M), \lfloor\cdot,\cdot\rfloor)$

Coefficient form: $\rho(e_a) = \rho_a^i(x)\mathbf{e}_i$ and $\lfloor e_a, e_b\rfloor = \mathbf{C}_{ab}^f(x)e_f$,
$\rho_a^i\mathbf{e}_i\rho_b^j - \rho_b^i\mathbf{e}_i\rho_a^j = \rho_f^j C_{ab}^f$ and $\sum_{\text{cycl } (a,b,f)} (\rho_a^i\partial_i C_{be}^f + C_{be}^d C_{ad}^f) = 0$

Example: Nonholonomic Lie algebroids: $\mathbf{E} = T\mathbf{V}$, for $\mathbf{V} = (V, \mathbf{N})$

slide 8: N–adapted prolongation Lie algebroid

Lie d–algebroid $\mathcal{E} = (\mathbf{E}, \lfloor\cdot,\cdot\rfloor, \rho)$ and fibration $\pi : \mathbf{P} \to M$ over the same manifold M, $u^{\underline{\alpha}} = (x^i, y^A) \in P$, particular $\mathbf{P} = \mathbf{E}$

anchor map $\rho : \mathbf{E} \to \mathbf{TM}$ and the tangent map $\mathbf{T}\pi : \mathbf{TP} \to \mathbf{TM}$, subset

$\mathcal{T}_s^E\mathbf{P} := \{(b, v) \in \mathbf{E}_x \times T_x\mathbf{P}; \rho(b) = T_p\pi(v); p \in \mathbf{P}_x, \pi(p) = x \in M\}$

Theor–Definition.

The the prolongation $\mathcal{T}^{\mathbf{E}}\mathbf{P} := \bigcup_{s\in S}\mathcal{T}_s^{\mathbf{E}}\mathbf{P}$ of a nonholonomic $\mathbf{E}$ over π is another Lie d–algebroid
$\overline{z} = z^a\mathcal{X}_a + v^A\mathcal{V}_A \in \mathcal{T}^{\mathbf{E}}\mathbf{P}$ $\rho^\pi(Z) = \rho^i_a Z^a\mathbf{e}_i + V^A\partial_A$
Lie brackets $\lfloor\mathcal{X}_a, \mathcal{X}_b\rfloor^\pi = C^f_{ab}\mathcal{X}_f,\ \lfloor\mathcal{X}_a, \mathcal{V}_B\rfloor^\pi = 0,\ \ \lfloor\mathcal{V}_A, \mathcal{V}_B\rfloor^\pi = 0$

$(\mathcal{X}^a, \mathcal{V}^B)$ the dual bases to $(\mathcal{X}_a, \mathcal{V}_A)$
differential calculus for N–adapted differential forms using
$dx^i = \rho^i_a\mathcal{X}^a$, for $d\mathcal{X}^f = -\frac{1}{2}C^f_{ab}\mathcal{X}^a\wedge\mathcal{X}^b$, and $dy^A = \mathcal{V}^A$, for $d\mathcal{V}^A = 0$

slide 9: N–connections on prolongation Lie algebroids

Def. a h–v–splitting $\mathcal{N}: \mathcal{T}^{\mathbf{E}}\mathbf{P} = h\mathcal{T}^{\mathbf{E}}\mathbf{P} \oplus v\mathcal{T}^{\mathbf{E}}\mathbf{P}$
Locally $\mathbf{N} = N_i^A(x^k, y^B) dx^i \otimes \partial_A$ and $\mathcal{N} = \mathcal{N}_a^A \mathcal{X}^a \otimes \mathcal{V}_A$.

structures on $T\mathbf{P}$ and $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ are compatible if $\mathcal{N}_a^A = N_i^A \rho_a^i$.
Using $\mathcal{N}_a^A$, generate sections $\delta_a := \mathcal{X}_a - \mathcal{N}_a^A \mathcal{V}_A$ as local basis of $h\mathcal{T}^{\mathbf{E}}\mathbf{P}$.

Corollary: $\mathcal{N}_a^A$ on $\mathcal{T}^E\mathbf{P}$ determines N–adapted frames

$$\mathbf{e}_{\overline{\alpha}} := \{\delta_a = \mathcal{X}_a - \mathcal{N}_a^C \mathcal{V}_C, \mathcal{V}_A\}, \mathbf{e}^{\overline{\beta}} := \{\mathcal{X}^a, \delta^B = \mathcal{V}^B + \mathcal{N}_c^B \mathcal{X}^c\}$$

Neigenhuis tensor ${}^hN$ of the operator h,

$$\begin{aligned} {}^hN(\cdot,\cdot) &= \lfloor h\cdot, h\cdot \rfloor^\pi - h\lfloor h\cdot, \cdot \rfloor^\pi - h\lfloor \cdot, h\cdot \rfloor^\pi + h^2 \lfloor h\cdot, h\cdot \rfloor^\pi \\ &= -\frac{1}{2}\Omega_{ab}^C \mathcal{X}^a \wedge \mathcal{X}^b \otimes \mathcal{V}_C, \\ \Omega_{ab}^C &= \delta_b \mathcal{N}_a^C - \delta_a \mathcal{N}_b^C + C_{ab}^f \mathcal{N}_f^C \end{aligned}$$

slide 10: Canonical structures on Lie d–algebroids

Def. d–connection, $\mathcal{D} = (h\mathcal{D}, v\mathcal{D})$, on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ is a linear connection preserving under parallelism h–v–splitting

Def. torsion and curvature

$$\mathcal{T}(\overline{x}, \overline{y}) := \mathcal{D}_{\overline{x}}\overline{y} - \mathcal{D}_{\overline{y}}\overline{x} + \lfloor \overline{x}, \overline{y} \rfloor^{\pi} \ \& \ \mathcal{R}(\overline{x}, \overline{y})\overline{z} := \left(\mathcal{D}_{\overline{x}}\, \mathcal{D}_{\overline{y}} - \ \mathcal{D}_{\overline{y}}\mathcal{D}_{\overline{x}} - \mathcal{D}_{\lfloor \overline{x}, \overline{y} \rfloor^{\pi}} \right) \overline{z}$$

sections $\overline{x}, \overline{y}, \overline{z}$ of $\mathcal{T}^{E}\mathbf{P}$, $\overline{z} = z^{\overline{\alpha}}\mathbf{e}_{\overline{\alpha}} = z^{a}\delta_{a} + z^{A}\mathcal{V}_{A}$, or $\overline{z} = h\overline{z} + v\overline{z}$.

absolute different for N–adapted $\mathbf{e}_{\overline{\alpha}} := \{\delta_{a}, \mathcal{V}_{A}\}$ and $\mathbf{e}^{\overline{\beta}} := \{\mathcal{X}^{\alpha}, \delta^{B}\}$
associating to $\mathcal{D}$ a d–connection 1–form $\Gamma^{\overline{\gamma}}_{\ \overline{\alpha}} := \Gamma^{\overline{\gamma}}_{\ \overline{\alpha}\overline{\beta}}\mathbf{e}^{\overline{\beta}}$
N–adapted coefficients $\mathcal{T} = \{\mathbf{T}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}\}$ and $\mathcal{R} = \{\mathbf{R}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\delta}}\}$

Prop–Definition. metric structure as a nondegenerate symmetric second rank tensor $\overline{\mathbf{g}} = \{\ \mathbf{g}_{\overline{\alpha}\overline{\beta}}\} = \overline{\mathbf{g}} = h\mathbf{g} \oplus v\mathbf{g}$

metric compatible data $(\overline{\mathbf{g}}, \mathcal{D})$, $\mathcal{Q} = \mathcal{D}\overline{\mathbf{g}} = 0$ for h–/ v–components.

slide 11: Canonical d–connection and distortions

On $\mathcal{T}^{\mathbf{E}}\mathbf{P}$, $\mathbf{g}_{\overline{\alpha}\overline{\beta}} \rightarrow$ the standard torsionless Levi–Civita connection ∇ (which is not N–adapted)

Theor. $\exists$ canonical d–connection $\widehat{\mathcal{D}} = h\widehat{\mathcal{D}} + v\widehat{\mathcal{D}}$ completely defined by data $(\mathcal{N}, \mathbf{g}_{\overline{\alpha}\overline{\beta}})$ for which $\widehat{\mathcal{D}}\overline{\mathbf{g}} = 0$
and zero h- and v-torsions of $\widehat{\mathcal{T}}$: $\widehat{T}^{a}_{\ bf} = C^{a}_{\ bf}$ and $\widehat{T}^{A}_{\ BC} = 0$.

Remarks:

1) There is a canonical distortion relation $\widehat{\mathcal{D}} = \overline{\nabla} + \widehat{\mathcal{Z}}$
2) $h\widehat{\mathcal{T}}^{\alpha} = 0$ for $\widehat{\mathbf{D}}$ on **TM** but $h\widehat{\mathcal{T}}^{\alpha} \neq 0$ for $\widehat{\mathcal{D}}$ on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$, $\widehat{T}^{a}_{\ bf} = C^{a}_{\ bf}$

Nonholonomic deformations:

$$^{c}\widehat{\mathcal{D}} := \nabla + {}^{c}\widehat{\mathcal{Z}} : \ h\, {}^{c}\widehat{\mathcal{T}}^{\alpha} = 0 \text{ and } v\, {}^{c}\widehat{\mathcal{T}}^{\alpha} = 0.$$

slide 12: Canonical N–connection and almost symplectic structures

Theor. $\forall \mathcal{L} \in C^{\infty}(\mathbf{E})$ a canonical $\widetilde{\mathcal{N}} = \{\widetilde{\mathcal{N}}_a^f = -\frac{1}{2}(\partial_a \varphi^f + y^b C_{ba}^f)\}$ determined by semi–spray configurations encoding the solutions of the Euler–Lagrange equations

Prop. $\forall\ \overline{\mathbf{g}} = h\tilde{\mathbf{g}} \oplus v\tilde{\mathbf{g}},\ \widetilde{\mathbf{g}} := \tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}} \mathbf{e}^{\overline{\beta}} \otimes \mathbf{e}^{\overline{\beta}} = \tilde{g}_{ab}\, \mathcal{X}^a \otimes \mathcal{X}^b + \ \tilde{g}_{ab}\, \widetilde{\delta}^a \otimes \widetilde{\delta}^b$

$\widetilde{\mathbf{e}}_{\overline{\alpha}} := \{\widetilde{\delta}_a = \mathcal{X}_a - \widetilde{\mathcal{N}}_a^f \mathcal{V}_f, \mathcal{V}_b\}$ and $\widetilde{\mathbf{e}}^{\overline{\beta}} := \{\mathcal{X}^a, \widetilde{\delta}^b = \mathcal{V}^b + \widetilde{\mathcal{N}}_f^b \mathcal{X}^f\}$

$\overline{\mathbf{g}} = \{\overline{\mathbf{g}}_{\overline{\alpha}'\overline{\beta}'}\}, \mathbf{e}_{\overline{\gamma}'} = e^{\overline{\gamma}}_{\ \overline{\gamma}'} \mathbf{e}_{\overline{\gamma}}$ when $\overline{\mathbf{g}}_{\overline{\alpha}'\overline{\beta}'} = e^{\overline{\alpha}}_{\ \overline{\alpha}'} e^{\overline{\beta}}_{\ \overline{\beta}'} \tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}}$

Riemann–Lagrange almost symplectic structures

Prop.–Def. $\forall\ \mathcal{L} \rightarrow \widetilde{\mathcal{N}}$ canonical almost complex structure on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ following formulas $\widetilde{\mathcal{J}}(\widetilde{\mathbf{e}}_a) = -\mathcal{V}_{m+a}$ and $\widetilde{\mathcal{J}}(\mathcal{V}_{m+a}) = \widetilde{\mathbf{e}}_a,\ \widetilde{\mathcal{J}} \circ \widetilde{\mathcal{J}} = -\mathbb{I}$

d–tensor $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}^{\overline{\alpha}}_{\ \overline{\beta}} \widetilde{\mathbf{e}}_{\overline{\alpha}} \otimes \widetilde{\mathbf{e}}^{\overline{\alpha}} = -\mathcal{V}_{m+a} \otimes \mathcal{X}^a + \widetilde{\mathbf{e}}_a \otimes \widetilde{\delta}^a,\ \widetilde{\mathcal{J}}^{\overline{\alpha}}_{\ \overline{\beta}} = e^{\overline{\alpha}}_{\ \overline{\alpha}'} e^{\ \overline{\beta}'}_{\overline{\beta}}\, \widetilde{\mathcal{J}}^{\overline{\alpha}'}_{\ \overline{\beta}'}.$

Nijenhuis $^{\mathcal{J}}\Omega(\overline{x}, \overline{y}) := -[\overline{x}, \overline{y}] + [\mathcal{J}\overline{x}, \mathcal{J}\overline{y}] - \mathcal{J}[\mathcal{J}\overline{x}, \overline{y}] - \mathcal{J}[\overline{x}, \mathcal{J}\overline{y}]$

for any sections $\overline{x}, \overline{y}$ of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$.

slide 13: Definition of prolongation almost Kähler d–algebroids

almost Hermitian and Kähler d–algebroids

a) is defined by a triple $\mathcal{H}^{\mathbf{E}}\mathbf{E} = (\mathcal{T}^{\mathbf{E}}\mathbf{E}, \theta, \mathcal{J})$, where $\theta(\overline{x}, \overline{y}) := \mathbf{g}(\mathcal{J}\overline{x}, \overline{y})$
b) A prolong Lie d–algebr $\mathcal{H}^{\mathbf{E}}\mathbf{E}$ is $\mathcal{K}^{\mathbf{E}}\mathbf{E}$, if and only if $d\theta = 0$

For effective regular Lagrange configurations,

Theor. Having chosen $\mathcal{L}$, model equivalently a $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ as a $\mathcal{K}^{\mathbf{E}}\mathbf{E}$.

Proof. For $(\overline{\mathbf{g}} = \tilde{\mathbf{g}}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{J}}), \tilde{\theta}(\overline{x},\overline{y}) := \tilde{\mathbf{g}}(\mathcal{J}\overline{x},\overline{y})$
sections $\overline{x}, \overline{y}$ of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, N–adapted $\tilde{\theta} = \tilde{g}_{ab}\delta^a \wedge \mathcal{X}^b$, $\theta_{\overline{\alpha}'\overline{\beta}'} = e^{\overline{\alpha}}_{\ \overline{\alpha}'} e^{\overline{\beta}}_{\ \overline{\beta}'} \tilde{\theta}_{\overline{\alpha}\overline{\beta}}$
Let $\tilde{\omega} := \frac{1}{2}\frac{\partial \mathcal{L}}{\partial y^{m+a}}\mathcal{X}^a \ \rightarrow \ \tilde{\theta} = d\tilde{\omega}$ and $d\tilde{\theta} = dd\tilde{\omega} = 0$

$$\tilde{\theta} = \frac{1}{2}\tilde{\theta}_{ab}(x^i, y^C)\mathcal{X}^a \wedge \mathcal{X}^b + \frac{1}{2}\tilde{\theta}_{AB}(x^i, y^C)\tilde{\delta}^A \wedge \tilde{\delta}^B$$

slide 14: The canonical almost symplectic d–connection

Def. A metric compatible almost symplectic d–connection on $\mathcal{H}^{\mathbf{E}}\mathbf{E}$ of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, ${}^{\theta}\mathcal{D}_{\overline{x}}\theta = 0, \forall$ section $\overline{x}$ of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$.

Lemma: fix ${}^{\circ}\mathcal{D}$ on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ and construct almost symplectic ${}^{\theta}\mathcal{D}$

Theor. On $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, $\exists$ unique normal d–connection

$$ {}^{n}\mathcal{D} = \{ h\, {}^{n}\mathcal{D} = ({}^{n}_{h}\mathcal{D}_a = \widehat{\mathcal{D}}_a,\ {}^{n}_{v}\mathcal{D}_a = \widehat{\mathcal{D}}_a); v\, {}^{n}\mathcal{D} = ({}^{n}_{h}\mathcal{D}_A = \widehat{\mathcal{D}}_A,\ {}^{n}_{v}\mathcal{D}_A = \widehat{\mathcal{D}}_A) \} $$

$\widehat{\mathcal{D}}_a \tilde{\mathbf{g}}_{bc} = 0$ and $\widehat{\mathcal{D}}_A \tilde{\mathbf{g}}_{BC} = 0$, completely defined by $\overline{\mathbf{g}} = \tilde{\mathbf{g}}$ and $\mathcal{L}(x,y)$

Theor. ${}^{n}\mathcal{D} = \widetilde{\mathcal{D}}$ defines a unique almost symplectic d–connection, $\widetilde{\mathcal{D}} \equiv {}^{\theta}\widetilde{\mathcal{D}}$, ${}^{\theta}\widetilde{\mathcal{D}}\tilde{\theta}{=}0$ and $\widetilde{\mathbf{T}}^{a}{}_{cb} = \widetilde{\mathbf{T}}^{A}{}_{CB} = 0$

Concl. $\left[\overline{\mathbf{g}}, \mathcal{N}, \widehat{\mathcal{D}} = \overline{\nabla} + \widehat{\mathcal{Z}} \right] \approx \left[\tilde{\mathbf{g}}, \mathcal{L}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}} \right] \approx \left[\tilde{\theta}(\cdot,\cdot) := \tilde{\mathbf{g}}(\widetilde{\mathcal{J}}\,\cdot\,,\cdot),\ {}^{\theta}\widetilde{\mathcal{D}} \right]$

The Lie algebroid structure functions $(\rho^{i}_{a}, C^{f}_{ab})$ are encoded into nonholonomic distributions on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$

slide 15: almost Kähler Einstein and Lagrange Lie d–algebroids

Corollary–Definition The Ricci tensor of $\mathcal{D}$ on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ with $\overline{\mathbf{g}}$ is $\mathcal{R}ic = \{\mathbf{R}_{\overline{\alpha}\overline{\beta}} := \mathbf{R}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}\overline{\gamma}}\}$

N–adapted coefficients for Riemannian d–tensor $\mathcal{R}^{\overline{\alpha}}_{\ \overline{\beta}} = \{\mathbf{R}^{\overline{\alpha}}{}_{\overline{\beta}\overline{\gamma}\overline{\delta}}\}$

$R_{\overline{\alpha}\overline{\beta}} = \{R_{ab} := R^{c}{}_{abc},\ R_{aA} := -R^{c}{}_{acA},\ R_{Aa} := R^{B}{}_{AaB},\ R_{AB} := R^{C}{}_{ABC}\}$

The scalar curvature $\ {}^{s}\mathbf{R} := \mathbf{g}^{\overline{\alpha}\overline{\beta}}\mathbf{R}_{\overline{\alpha}\overline{\beta}} = \mathbf{g}^{ab}\mathbf{R}_{ab} + \mathbf{g}^{AB}\mathbf{R}_{AB}.$

the Einstein d–tensor $\mathbf{E}_{\overline{\alpha}\overline{\beta}} := \mathbf{R}_{\overline{\alpha}\overline{\beta}} - \frac{1}{2}\mathbf{g}_{\overline{\alpha}\overline{\beta}}\ {}^{s}\mathbf{R}$

on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, or $\mathcal{K}^{\mathbf{E}}\mathbf{E}$, respectively, for $\widehat{\mathcal{D}}$ and $\widetilde{\mathcal{D}} = {}^{\theta}\widetilde{\mathcal{D}}$.

Prescribing $(\mathcal{L}; \rho^{i}_{a}, C^{f}_{ab})$, $\widehat{\mathcal{R}}ic = \lambda\overline{\mathbf{g}}$, $\widehat{\mathcal{Z}} = 0$

almost Kähler variables, $\widetilde{\mathcal{R}}ic = \lambda\mathbf{g}$, $\widetilde{\mathcal{Z}} = 0$

slide 16: almost Kähler – Ricci Evolution and Lie Algebroids

Perelman's functionals in almost Kähler variables and $\mathcal{K}^{\mathbf{E}}\mathbf{E}$

Models of non–Riemannian Ricci flow evolution:

- nonholonomic, Lagrange–Finsler flows, nonsymmetric metrics
- noncommutative Ricci flows; almost Kähler and DQ flows
- fractional derivative and diffusion evoluton
- Lagrange–Ricci flows on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$; Flows for $\mathcal{K}^{\mathbf{E}}\mathbf{E}$

Remark: proofs for $[\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}, \mathcal{L}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}}] \approx [\tilde{\theta}(\cdot,\cdot) := \tilde{\mathbf{g}}(\widetilde{\mathcal{J}}\cdot,\cdot),\ {}^{\theta}\widetilde{\mathcal{D}} = \overline{\nabla} + \widetilde{\mathcal{Z}}]$

Lemma: Perelman's functionals equivalently in canon. almost Kähler form

$$\widetilde{\mathcal{F}}(\widetilde{\mathbf{g}}, \widetilde{\mathcal{D}}, \check{f}) = \int_{\overline{\mathcal{V}}} ({}^{s}\widetilde{\mathbf{R}} + |h\widetilde{\mathcal{D}}\check{f}|^2 + |\ v\widetilde{\mathcal{D}}\check{f}|)^2) e^{-\check{f}}\, dv,$$

$$\widetilde{\mathcal{W}}(\widetilde{\mathbf{g}}, \widetilde{\mathcal{D}}, \check{f}, \check{\tau}) = \int_{\overline{\mathcal{V}}} [\check{\tau}({}^{s}\widetilde{\mathbf{R}} + |h\widetilde{\mathcal{D}}\check{f}| + |v\widetilde{\mathcal{D}}\check{f}|)^2 + \check{f} - 2m]\check{\mu} dv$$

slide 17: N–adapted metric and almost symplectic evolution eqs

Theorem: a) $\widetilde{\mathcal{D}}$ preserving a symmetric metric structure $\tilde{\mathbf{g}}$ on $\mathcal{T}^E\mathbf{E}$

$$\begin{aligned}\frac{\partial \widetilde{\mathbf{g}}_{ab}}{\partial \chi} &= -(\widetilde{\mathbf{R}}_{ab} + \widetilde{\mathbf{Z}}ic_{ab}),\ \frac{\partial \widetilde{\mathbf{g}}_{AB}}{\partial \chi} = -(\widetilde{\mathbf{R}}_{AB} + \widetilde{\mathbf{Z}}ic_{AB}),\\ \widetilde{\mathbf{R}}\ _{aA} &= -\widetilde{\mathbf{Z}}ic_{aA},\ \widetilde{\mathbf{R}}\ _{Aa} = \widetilde{\mathbf{Z}}ic_{Aa},\\ \frac{\partial \widetilde{f}}{\partial \chi} &= -(\widetilde{\Delta} + {}^{Z}\widetilde{\Delta})\widetilde{f} + \left|\left(\widetilde{\mathcal{D}} - \widetilde{\mathcal{Z}}\right)\widetilde{f}\right|^2 - {}^{s}\widetilde{\mathbf{R}} - {}^{s}\widetilde{\mathbf{Z}},\end{aligned}$$

$$\frac{\partial}{\partial \chi}\mathcal{F}(\widetilde{\mathbf{g}}, \widetilde{\mathcal{D}}, \widetilde{f}) = \int_{\widetilde{\mathcal{V}}}[|\widetilde{\mathbf{R}}_{ab} + \widetilde{\mathbf{Z}}ic_{ab} + (\widetilde{\mathcal{D}}_a - \widetilde{\mathcal{Z}}_a)(\widetilde{\mathcal{D}}_b - \widetilde{\mathcal{Z}}_b)\widetilde{f}|^2$$
$$+|\widetilde{\mathbf{R}}_{AB} + \widetilde{\mathbf{Z}}ic_{AB} + (\widetilde{\mathcal{D}}_A - \widetilde{\mathcal{Z}}_A)(\widetilde{\mathcal{D}}_B - \widetilde{\mathcal{Z}}_B)\widetilde{f}|^2]e^{-\widetilde{f}}dv,\ \int_{\widetilde{\mathcal{V}}} e^{-\widetilde{f}}dv = const.$$

b) $\tilde{\theta} = \tilde{g}_{ab}\delta^a \wedge \mathcal{X}^b$ and $\mathcal{K}^{\mathbf{E}}\mathbf{E}$, $\frac{\partial \tilde{\theta}_{ab}}{\partial \chi} = -\widetilde{\mathbf{R}}_{[ab]}$, $\frac{\partial \tilde{\theta}_{AB}}{\partial \chi} = -\widetilde{\mathbf{R}}_{[AB]}$

On $\mathcal{T}^E\mathbf{E}$ with $\widehat{\mathcal{D}}$, $\frac{\partial \widetilde{\mathbf{g}}_{ab}}{\partial \chi} = -2\widehat{\mathbf{R}}_{ab}$, $\frac{\partial \widetilde{\mathbf{g}}_{AB}}{\partial \chi} = -2\widehat{\mathbf{R}}_{AB}$,

$\widehat{\mathbf{R}}\ _{aA} = 0,\ \widehat{\mathbf{R}}\ _{Aa} = 0,\ \frac{\partial \widehat{f}}{\partial \chi} = -\widehat{\Delta}\widehat{f} + \left|\widehat{\mathcal{D}}\widehat{f}\right|^2 - {}^{s}\widehat{\mathbf{R}}$

slide 18: Geometric thermodynamics of almost Kähler d–algebroids

Theor. The Ricci flow evolution eqs with symmetric metrics and respective almost symplectic forms on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ and $\mathcal{K}^{\mathbf{E}}\mathbf{E}$ are solutions of eqs with the Ricci d–tensor (previous slide) and

$$\frac{\partial \tilde{f}}{\partial \chi} = -(\widetilde{\Delta} + {}^{Z}\tilde{\Delta})\tilde{f} + \left|(\widetilde{\mathcal{D}}_a - \widetilde{\mathcal{Z}}_a)\tilde{f}\right|^2 - \ {}^{s}\widetilde{\mathbf{R}} + \frac{2m}{\hat{\tau}},$$

$\frac{\partial \tilde{\tau}}{\partial \chi} = -1$, and conditions for the "minus entropy":

$\frac{\partial}{\partial \chi}\tilde{\mathcal{W}}(\widetilde{\mathbf{g}}(\chi), \tilde{f}(\chi), \tilde{\tau}(\chi)) = 2\int_{\mathcal{V}} \tilde{\tau}[|\widetilde{\mathbf{R}}_{\overline{\alpha}\overline{\beta}} - \widetilde{\mathbf{Z}}ic_{\overline{\alpha}\overline{\beta}}$
$+(\widetilde{\mathcal{D}}_{\overline{\alpha}} - \widetilde{\mathcal{Z}}_{\overline{\alpha}})(\widetilde{\mathcal{D}}_{\overline{\beta}} - \widetilde{\mathcal{Z}}_{\overline{\beta}})\tilde{f} - \frac{1}{2\tilde{\tau}}\tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}}|^2](4\pi\tilde{\tau})^{-m} e^{-\tilde{f}} dv,$

$\int_{\mathcal{V}} e^{-\tilde{f}} dv = const.$

slide 19: Thermodynamical values

Remarks: stochastic processes, diffusion, fractional calculus, quantum entropy....

Theor. a) canonical thermodynamic values on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$,

$\left\langle \widehat{E} \right\rangle = -\widehat{\tau}^2 \int_{\mathcal{V}} (\ {}^s\widehat{\mathbf{R}} + |\widehat{\mathcal{D}}\widehat{f}|^2 - \frac{m}{\widehat{\tau}})\widehat{\mu}\, dv, \widehat{S} = -\int_{\mathcal{V}} [\widehat{\tau}(\ {}^s\widehat{\mathbf{R}} + |\widehat{\mathcal{D}}\widehat{f}|^2) + \widehat{f} - 2m]\widehat{\mu}\, dv$

$\widehat{\sigma} = 2\,\widehat{\tau}^4 \int_{\mathcal{V}} [|\widehat{\mathbf{R}}_{\overline{\alpha}\overline{\beta}} - \widehat{\mathbf{Z}}ic_{\overline{\alpha}\overline{\beta}} + (\widetilde{\mathcal{D}}_{\overline{\alpha}} - \widehat{\mathcal{Z}}_{\overline{\alpha}})(\widehat{\mathcal{D}}_{\overline{\beta}} - \widehat{\mathcal{Z}}_{\overline{\beta}})\widehat{f} - \frac{1}{2\widehat{\tau}}\mathbf{g}_{\overline{\alpha}\overline{\beta}}|^2]\widehat{\mu}\, dv$

b) and/or by effective Lagrange and/or almost Kähler Ricci flows on $\mathcal{K}^{\mathbf{E}}\mathbf{E}$,

$$\left\langle \tilde{E} \right\rangle = -\tilde{\tau}^2 \int_{\widetilde{\mathcal{V}}} (\ {}^s\widetilde{\mathbf{R}} + |\widetilde{\mathcal{D}}\tilde{f}|^2 - \frac{m}{\tilde{\tau}})\tilde{\mu}\, dv, \tilde{S} = -\int_{\widetilde{\mathcal{V}}} [\tilde{\tau}(\ {}^s\widetilde{\mathbf{R}} + |\widetilde{\mathcal{D}}\tilde{f}|^2) + \tilde{f} - 2m]\tilde{\mu}\, dv$$

$$\tilde{\sigma} = 2\,\tilde{\tau}^4 \int_{\widetilde{\mathcal{V}}} [|\widetilde{\mathbf{R}}_{\overline{\alpha}\overline{\beta}} + \widetilde{\mathcal{D}}_{\overline{\alpha}}\widetilde{\mathcal{D}}_{\overline{\beta}}\tilde{f} - \frac{1}{2\tilde{\tau}}\tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}}|^2]\tilde{\mu}\, dv$$

Proof: using $\tilde{Z} = \exp\left\{ \int_{\widetilde{\mathcal{V}}} [-\tilde{f} + m]\ \tilde{\mu} dv \right\}$ on $\mathcal{K}^{\mathbf{E}}\mathbf{E}$, $\nabla \to \widehat{\mathcal{D}}$, or $\nabla \to \widetilde{\mathcal{D}}$

partition funct $Z = \int \exp(-\beta E) d\omega(E)$ for a canonical ansamble at temperature β^{-1}; temperature is defined by the measure determined by the density of states $\omega(E)$; statistical analogy computing thermodynamical values: $\langle E \rangle := -\partial \log Z/\partial\beta$, entropy $S := \beta \langle E \rangle + \log Z$ and the fluctuation $\sigma := \left\langle (E - \langle E \rangle)^2 \right\rangle = \partial^2 \log Z/\partial\beta^2$

slide 20: almost Kähler Solitons with Lie Algebroid Symmetries

Preliminaries on Lie d–algebroid solitons

Def. The geometric data $[\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}, \mathcal{L}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}}] \approx [\tilde{\theta}(\cdot,\cdot) := \tilde{\mathbf{g}}(\widetilde{\mathcal{J}}\cdot,\cdot), \ {}^{\theta}\widetilde{\mathcal{D}} = \overline{\nabla} + \widetilde{\mathcal{Z}}]$ for a complete Riemannian metric $\overline{\mathbf{g}}$ on a smooth $\mathcal{T}^{\mathbf{E}}\mathbf{E}, \mathcal{K}^{\mathbf{E}}\mathbf{E}$ define a gradient almost Kähler–Ricci d–algebroid soliton if $\exists$ a smooth potential function $\tilde{\kappa}(x^i, y^C)$

$$\widetilde{\mathbf{R}}_{\ \overline{\beta}\overline{\gamma}} + \widetilde{\mathcal{D}}_{\overline{\beta}}\widetilde{\mathcal{D}}_{\overline{\gamma}}\tilde{\kappa} = \lambda \tilde{\mathbf{g}}_{\ \overline{\beta}\overline{\gamma}},$$

$$\text{equivalently, } {}^{\theta}\widetilde{\mathbf{R}}_{\ [\overline{\beta}\overline{\gamma}]} + {}^{\theta}\widetilde{\mathcal{D}}_{[\overline{\beta}}\ {}^{\theta}\widetilde{\mathcal{D}}_{\overline{\gamma}]}\tilde{\kappa} = {}^{\lambda}\tilde{\theta}_{\ \overline{\beta}\overline{\gamma}},$$

$\exists$ three types: $\lambda = const$: steady ones, for $\lambda = 0$; shrinking, for $\lambda > 0$; and expanding, for $\lambda < 0$.

Prop. Let $(\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}, \mathcal{L}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}}; \tilde{\kappa})$ be a complete shrinking soliton on $\mathcal{T}^{\mathbf{E}}\mathbf{E}, \mathcal{K}^{\mathbf{E}}\mathbf{E}$. Using nonholonomic frame deformations, redefined $\widehat{\kappa}(x^i, y^C)$, for $\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}$,

$$\widehat{\mathbf{R}}_{\ \overline{\beta}\overline{\gamma}} + \widehat{\mathcal{D}}_{\overline{\beta}}\widehat{\mathcal{D}}_{\overline{\gamma}}\widehat{\kappa} = \lambda \overline{\mathbf{g}}_{\ \overline{\beta}\overline{\gamma}}$$

$\widehat{\mathcal{Z}} = 0$ and/or $\widetilde{\mathcal{Z}} = 0$ result in the Levi–Civita configurations.

slide 21: Generalized Einstein eqs encoding Lie d–algebroid structures

$\widehat{\mathcal{D}}_{\overline{\gamma}}\widehat{\kappa} = \mathbf{e}_{\overline{\gamma}}\widehat{\kappa} = \kappa_{\overline{\gamma}} = const$, i.e. $\delta_a \widehat{\kappa} = \mathcal{X}_a \widehat{\kappa} - \mathcal{N}_a^C \kappa_C = 0$ and $\mathcal{V}_A \widehat{\kappa} = \kappa_A$.

$\mathbf{E} = \mathbf{P}$ with $2+2$ splitting, $a, b, ... = 1, 2; i', j', ... = 1, 2$ and $A, B, ... = 3, 4$.
$u^\mu = (x^i, y^a) = (x^1, x^2, y^3, y^4)$, on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, $\mathbf{g} \rightarrow \overline{\mathbf{g}}$,

Prime metric

$$\begin{aligned} \mathbf{g} &= \mathring{g}_\alpha(u)\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = \mathring{g}_i(x)dx^i \otimes dx^i + \mathring{h}_a(x,y)\mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^\alpha &= (dx^i, \mathbf{e}^a = dy^a + \mathring{N}_i^a(u)dx^i), \\ \mathbf{e}_\alpha &= (\mathbf{e}_i = \partial/\partial y^a - \mathring{N}_i^b(u)\partial/\partial y^b,\ e_a = \partial/\partial y^a). \end{aligned}$$

$h^* := \partial_3$ and $\mathcal{N}_a^3 = w_a(x^k, y^3), \mathcal{N}_a^4 = n_a(x^k, y^3)$.

Target metric

$$\begin{aligned} \overline{\mathbf{g}} &= \mathbf{g}_{\overline{\alpha}\overline{\beta}}\mathbf{e}^{\overline{\beta}} \otimes \mathbf{e}^{\overline{\beta}} = \mathbf{g}_a\, \mathcal{X}^a \otimes \mathcal{X}^a + \mathbf{g}_A\, \delta^A \otimes \delta^A \\ &= \eta_a(x^k)\mathring{g}_a \mathcal{X}^a \otimes \mathcal{X}^a + \eta_A(x^k, y^3)\mathring{h}_A \delta^A \otimes \delta^A \end{aligned}$$

slide 22:

Propos. $\partial_a \to \mathcal{X}_a$ and $\mathcal{V}_A = \partial_A$

$$-\widehat{\mathbf{R}}^1_1 = -\widehat{\mathbf{R}}^2_2 = \frac{1}{2g_1 g_2}[\mathcal{X}_1(\mathcal{X}_1 g_2) - \frac{\mathcal{X}_1 g_1 \ \mathcal{X}_1 g_2}{2g_1} - \frac{(\mathcal{X}_1 g_2)^2}{2g_2} + \mathcal{X}_2(\mathcal{X}_2 g_1) - \frac{\mathcal{X}_2 g_1 \ \mathcal{X}_2 g_2}{2g_2} - \frac{(\mathcal{X}_2 g_1)^2}{2g_1}] = \lambda,$$

$$-\widehat{\mathbf{R}}^3_3 = -\widehat{\mathbf{R}}^4_4 = \frac{1}{2h_3 h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] = \lambda,$$

$$\widehat{\mathbf{R}}_{3a} = \frac{w_a}{2h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] + \frac{h_4^*}{4h_4}(\frac{\mathcal{X}_a h_3}{h_3} + \frac{\mathcal{X}_a h_4}{h_4}) - \frac{\mathcal{X}_a h_4^*}{2h_4} = 0,$$

$$\widehat{\mathbf{R}}_{4a} = \frac{h_4}{2h_3} n_a^{**} + (\frac{h_4}{h_3} h_3^* - \frac{3}{2} h_4^*)\frac{n_a^*}{2h_3} = 0;$$

for the potential function $\mathcal{X}_a \widehat{\kappa} - w_a \kappa_3 - n_a \kappa_4 = 0$ and $\mathcal{V}_A \widehat{\kappa} = \kappa_A$,
LC conditions $\widehat{\mathcal{Z}} = 0$;
$w_a^* = (\mathcal{X}_a - w_a \partial_3) \ln \sqrt{|h_3|}, (\mathcal{X}_a - w_a \partial_3) \ln \sqrt{|h_4|} = 0, \mathcal{X}_b w_a = \mathcal{X}_a w_b, n_a^* = 0, \partial_a n_b = \partial_b n_a.$
a nontrivial source λ, $g_a = \epsilon_a e^{\psi(x^k)}$, $\epsilon_a = \pm 1$ and $h_a^* \neq 0$,

slide 23: Generating off–diagonal solutions

Theor. PDEs decouple in N–adapted form,

$$\begin{aligned} \epsilon_1 \mathcal{X}_1(\mathcal{X}_1 \psi) + \epsilon_2 \mathcal{X}_2(\mathcal{X}_2 \psi) &= 2\,\lambda \\ \phi^* h_4^* &= 2 h_3 h_4 \lambda \\ \beta w_A - \alpha_A &= 0, \\ n_A^{**} + \gamma n_A^* &= 0, \end{aligned}$$

$$\text{for } \alpha_A = h_4^* \partial_A \phi, \beta = h_4^*\ \phi^*, \gamma = \left(\ln |h_4|^{3/2}/|h_3|\right)^*,$$

$$\text{generating function } \phi = \ln |h_4^*/\sqrt{|h_3 h_4|}|$$

LC–solutions:

$$ds^2 = e^{\psi(x^k)}[\epsilon_1(\mathcal{X}^1)^2 + \epsilon_2(\mathcal{X}^2)^2] + \epsilon_3 \frac{(\check{\Phi}^*)^2}{\lambda \check{\Phi}^2}[\mathcal{V}^3 + (\mathcal{X}_a \widetilde{A}[\check{\Phi}])\mathcal{X}^a]^2 + \epsilon_4 \frac{\check{\Phi}^2}{4|\lambda|}[\mathcal{V}^4 + (\mathcal{X}_a n)\mathcal{X}^a]^2$$

the solutions defining Ricci solitons can be with nontrivial torsion.
Remark: nonholonomically induced torsion,

$$ds^2 = e^{\psi(x^k)}[\epsilon_1(\mathcal{X}^1)^2 + \epsilon_2(\mathcal{X}^2)^2] + \epsilon_3(z_3)^2[\mathcal{V}^3 + \frac{\mathcal{X}_a \Phi}{\Phi^*}\mathcal{X}^a]^2 + \epsilon_4(z_4)^2[\mathcal{V}^4 + (\,_1n_a + \,_2n_a \int dy^3 \frac{(z_3)^2}{(z_4)^3})\mathcal{X}^a]^2$$

where the values $z_3(x^k, y^3)$ and $z_4(x^k, y^3)$

page 24: Nonholonomic spinors and Dirac operators

Clifford d–algebra, $\wedge V^{n+m}$ algebra, product $\mathbf{uv} + \mathbf{vu} = 2\mathbf{g}(\mathbf{u}, \mathbf{v})\ \mathbb{I}$;

$$ {}^{h}u\, {}^{h}v + {}^{h}v\, {}^{h}u = 2\ {}^{h}g(u,v)\ {}^{h}\mathbb{I},\ ,\ {}^{v}u\ \ {}^{v}v + {}^{v}v\, {}^{v}u = 2\ {}^{v}h({}^{v}u, {}^{v}v)\ {}^{v}\mathbb{I}, $$

$\mathbf{u} = ({}^{h}u, {}^{v}u)$, $\mathbf{v} = ({}^{h}v, {}^{v}v) \in V^{n+m}$, $\mathbb{I}$, ${}^{h}\mathbb{I}$ and ${}^{v}\mathbb{I}$ are unity matrices $(n+m) \times (n+m)$, or $n \times n$ and $m \times m$.

A metric ${}^{h}g$ on $h\mathbf{V}$ is defined by sections of $Th\mathbf{V}$ provided with a bilinear symmetric form on continuous sections $\Gamma(T\ h\mathbf{V})$.

Clifford h–algebras ${}^{h}\mathcal{C}l(T_x h\mathbf{V})$, in any point $x \in Th\mathbf{V}$,

$$ \gamma_i\gamma_j + \gamma_j\gamma_i = 2\ g_{ij}\ {}^{h}\mathbb{I}. $$

Definition: A Clifford d–space on $\mathbf{V}$, with $\mathbf{g}(x,y)$ and $\mathbf{N}$ is a Clifford bundle $\mathcal{C}l(\mathbf{V}) = {}^{h}\mathcal{C}l(h\mathbf{V}) \oplus {}^{v}\mathcal{C}l(v\mathbf{V})$, Clifford h–space ${}^{h}\mathcal{C}l(h\mathbf{V}) \doteq {}^{h}\mathcal{C}l(T^{*}h\mathbf{V})$, Clifford v–space ${}^{v}\mathcal{C}l(v\mathbf{V}) \doteq {}^{v}\mathcal{C}l(T^{*}v\mathbf{V})$.

page 25: (Almost Kähler) N–adapted Dirac operators

d–gamma matrix relations $\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}} + \gamma^{\hat{\beta}}\gamma^{\hat{\alpha}} = 2\delta^{\hat{\alpha}\hat{\beta}}\ \mathbb{I}$,
action of $du^{\alpha} \in \mathcal{C}l(\mathbf{V})$ on a d–spinor $\breve{\psi} \in\ \mathbf{S},\ \mathbf{c}(du^{\hat{\alpha}}) \doteq \gamma^{\hat{\alpha}}$,
$\mathbf{c} = (du^{\alpha})\ \breve{\psi} \doteq \gamma^{\alpha}\ \breve{\psi} \equiv e^{\alpha}_{\ \hat{\alpha}}\ \gamma^{\hat{\alpha}}\ \breve{\psi}$,

$$\gamma^{\alpha}(u)\gamma^{\beta}(u) + \gamma^{\beta}(u)\gamma^{\alpha}(u) = 2g^{\alpha\beta}(u)\ \mathbb{I}.$$

Canon. spin Cartan d–con.: ${}^{\theta}_{\mathbf{S}}\widehat{\nabla} \doteq\ {}^{L}\delta - \frac{1}{4}\ {}^{\theta}\widehat{\Gamma}^{\alpha}_{\ \beta\mu}\gamma_{\alpha}\gamma^{\beta}\ \delta u^{\mu}$.

Definition: The Dirac d–operator (h–operator) on a spin N–anholonomic manifold $(\mathbf{V}, \mathbf{S}, J)$ (h–spin manifold $(h\mathbf{V},\ {}^{h}S,\ {}^{h}J)$, or v–spin manifold $(v\mathbf{V},\ {}^{v}S,\ {}^{v}J)$) is

$$\mathbb{D} \doteq -i\ (\widehat{\mathbf{c}} \circ\ {}_{\mathbf{S}}\nabla) = \left(\ {}^{h}\mathbb{D} = -i\ (\ {}^{h}\widehat{c} \circ\ {}^{h}_{\mathbf{S}}\nabla),\ \ {}^{v}\mathbb{D} = -i\ (\ {}^{v}\widehat{c} \circ\ {}^{v}_{\mathbf{S}}\nabla)\right)$$

Dirac d–operators are called almost Kähler and denoted ${}_{\theta}\widehat{\mathbb{D}} = (\ {}^{h}_{\theta}\widehat{\mathbb{D}},\ {}^{v}_{\theta}\widehat{\mathbb{D}}\ )$ if defined for the Cartan/ normal d–connection.

page 26: The spectral action/functional paradigm:

Standard models, particles & "extracted" from noncommut. geometry,

spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, postulating action $Tr\ f(\mathcal{D}^2/\Lambda^2) + <\Psi|\mathcal{D}|\Psi>$. Tr is the trace in operator algebra, Ψ is a spinor, all defined for a Hilbert space $\mathcal{H}$, Λ is a cutoff scale and f is a positive function. Spectral action depends on spectrum of Dirac operator $\mathcal{D}$ on a space defined by a noncommutative associative algebra $\mathcal{A} = C^\infty(V) \otimes\ ^P\mathcal{A}$.

Spectral geometry of $\mathcal{A}$: product rule $\mathcal{H} = L^2(V, S) \otimes\ ^P\mathcal{H}$, Hilbert sp. L^2 spinors $L^2(V, S)$, Hilbert space of quarks and leptons $^P\mathcal{H}$ fixing the choice of the Dirac operator ^PD & action $^P\mathcal{A}$ for fund. particles. Dirac operator $\mathcal{D} =\ ^VD \otimes 1 + \gamma_5 \otimes\ ^PD$, Dirac operator ^VD of the Levi–Civita spin connection on V. Spectral functionals contain in commutative limit the Perelman's functionals for Ricci flows.

Scal prod on $\Gamma^\infty(\mathbf{S})$, $<\breve{\psi}, \breve{\phi}> \doteq \int_{\mathbf{V}} (\breve{\psi}|\breve{\phi})|\nu_{\mathbf{g}}|$, $\mathbf{V}:\ \nu_{\mathbf{g}} = \sqrt{det|g|\ det|h|}\ dx^1...dx^n\ dy^{n+1}...dy^{n+m}$
Hilbert d–space by completing $\Gamma^\infty(\mathbf{S})$, sc. pr.
$^N\mathcal{H} := L_2(\mathbf{V}, \mathbf{S}) = [\ ^h\mathcal{H} = L_2(h\mathbf{V},\ ^hS),\ ^v\mathcal{H} = L_2(v\mathbf{V},\ ^vS)]$

A canonical (almost Kähler) spectral d–triple $(\ ^N\mathcal{A},\ ^N\mathcal{H},\ _\theta\widehat{\mathbb{D}})$ for a d–algebra $^N\mathcal{A}$ is defined by 1) a Hilbert d–space $^N\mathcal{H}$, 2) a representation of $^N\mathcal{A}$ in the algebra $^N\mathcal{B}(\ ^N\mathcal{H})$ of d–operators bounded on $^N\mathcal{H}$, 3) by a self–adjoint d–operator $^N\mathcal{H}$, of compact resolution,(an operator D is of compact resolution if for any $\lambda \in sp(D)$ the operator $(D - \lambda\mathbb{I})^{-1}$ is compact) such that $[\ ^N\mathcal{H}, a] \in\ ^N\mathcal{B}(\ ^N\mathcal{H})$ for any $a \in\ ^N\mathcal{A}$.

page 27: Spectral triples and distance in d–spinor spaces

Theorem: (Distance) Let $({}^N\mathcal{A}, {}^N\mathcal{H}, {}_\theta\widehat{\mathbb{D}}, \mathbf{J}, {}_{[cr]}\Gamma)$ a noncom. geometry, irreducible for ${}^N\mathcal{A} \doteq C^\infty(\mathbf{V})$, where **V** is a compact, connected and oriented manifold without boundaries, of spectral dimension $dim\, \mathbf{V} = n + n$. There are satisfied:

1. $\exists$ a unique d–metric $\mathbf{g}({}_\theta\widehat{\mathbb{D}}) = ({}^h g, {}^v g)$, "nonlinear geodes." dist. on **V**, $d(u_1, u_2) = \sup_{f \in C(\mathbf{V})} \left\{ f(u_1, u_2) / \parallel [{}_\theta\widehat{\mathbb{D}}, f] \parallel \leq 1 \right\}$, $\forall$ smooth $f \in C(\mathbf{V})$.
2. An almost Kähler model of N–anholonomic manifold **V** is a spin N–anholonomic space, operators ${}_\theta\widehat{\mathbb{D}}'$ satisfying the condition $\mathbf{g}({}_\theta\widehat{\mathbb{D}}') = \mathbf{g}({}_\theta\widehat{\mathbb{D}})$ (and canonically derived almost Kähler spaces with ${}^L\theta({}_\theta\widehat{\mathbb{D}}') = {}^L\theta({}_\theta\widehat{\mathbb{D}}))$ define an union of affine spaces identified by the d–spinor structures on **V**.
3. The functional $\mathcal{S}({}_\theta\widehat{\mathbb{D}}) \doteq \int |{}_\theta\widehat{\mathbb{D}}|^{-n-n+2}$ defines a quadratic d–form with $(n+n)$–splitting for every affine space which is minimal for ${}_\theta\widehat{\mathbb{D}} = {}_\theta\overleftarrow{\mathbb{D}}$ as the canonical almost Kähler Dirac d–operator corresponding to the d–spin structure with the minimum proportional to the Einstein–Hilbert action for the canonical Cartan/ normal d–connection with d–scalar curv. ${}_\theta^s\mathbf{R}$,
$\mathcal{S}({}_\theta\overleftarrow{\mathbb{D}}) = -\frac{n-1}{12} \int_{\mathbf{V}} {}_\theta^s\mathbf{R} \sqrt{{}^h g} \sqrt{{}^v h}\, dx^1 ... dx^n\, \delta y^{n+1} ... \delta y^{n+n}.$

page 28: Spectral nonholonomic flows and Perelman functionals

Family of generalized d–operators

$$ {}_{\theta}\mathcal{D}^2(\chi) = -[\frac{\mathbb{I}}{2}\ {}^{L}\theta^{\alpha\beta}(\chi)[\ {}^{L}\mathbf{e}_{\alpha}(\chi)\ {}^{L}\mathbf{e}_{\beta}(\chi) - \ {}^{L}\mathbf{e}_{\beta}(\chi)\ {}^{L}\mathbf{e}_{\alpha}(\chi)] + \mathbf{A}^{\nu}(\chi)\ {}^{L}\mathbf{e}_{\nu}(\chi) + \mathbf{B}(\chi)] $$

$\chi \in [0, \chi_0)$, matrices $\mathbf{A}^{\nu}(\chi)$ and $\mathbf{B}(\chi)$ determined by ${}_{\theta}\mathbb{D}$ induced by ${}_{\theta}\mathbf{D}$; for the Cartan/ normal d–connection, ${}_{\theta}\widehat{\mathcal{D}}^2, \widehat{\mathbf{A}}^{\nu}$ and $\widehat{\mathbf{B}}$. We introduce functionals $\mathcal{F}$ and $\mathcal{W}$ depending on χ,

$$ \begin{aligned} \mathcal{F} &= Tr\left[\ {}^{1}f\ (\chi)(\ {}^{1\phi}\mathcal{D}^2(\chi)/\Lambda^2)\right] \simeq \sum_{k\geq 0}\ {}^{1}f_{(k)}(\chi)\ \ {}^{1}a_{(k)}(\ {}^{1\phi}\mathcal{D}^2(\chi)/\Lambda^2) \\ \mathcal{W} &= {}^{2}\mathcal{W} + {}^{3}\mathcal{W}, \\ \text{for } {}^{e}\mathcal{W} &= Tr\left[\ {}^{e}f(\chi)(\ {}^{e\phi}\mathcal{D}^2(\chi)/\Lambda^2)\right] = \sum_{k\geq 0}\ {}^{e}f_{(k)}(\chi)\ {}^{e}a_{(k)}(\ {}^{e\phi}\mathcal{D}^2(\chi)/\Lambda^2), \end{aligned} $$

cutting parameter Λ^2 for both cases $e = 2, 3$. Functions ${}^{b}f$, with label b taking values $1, 2, 3$. Coefficients computed as "N–adapted" Seeley – de Witt coefficients.

page 29: Main Theorems on "noncommutative" Perelman functionals

Theorem: *For the scaling factor* ${}^1\phi = -f/2$*, the first spectral functional* $\mathcal{F} = {}^P\mathcal{F}({}^L\theta, {}_\theta\mathbf{D}, f)$ *can be approximated as the first Perelman functional*

$$ {}^P\mathcal{F} = \int_{\mathbf{V}} \delta V\, e^{-f}[\, {}^\theta_s\mathbf{R}(e^{-f}\ {}^\mathcal{L}\theta_{\mu\nu}) + \frac{3}{2} e^{f}\ {}^\mathcal{L}\theta^{\alpha\beta}(\ {}^\mathcal{L}\mathbf{e}_\alpha f\ {}^\mathcal{L}\mathbf{e}_\beta f - \ {}^\mathcal{L}\mathbf{e}_\beta f\ {}^\mathcal{L}\mathbf{e}_\alpha f)]. $$

Theorem: *2d spectr.funct.* $\mathcal{W} = {}^P\mathcal{W}({}^L\theta, {}_\theta\mathbf{D}, f)$ *is approx. as 2d Perelman funct.*

$$ {}^P\mathcal{W} = \int_{\mathbf{V}} \delta V \mu \times [\chi(\, {}^\theta_s\mathbf{R}(e^{-f}\ {}^L\theta_{\mu\nu}) + \frac{3}{2} e^{f}\ {}^L\theta^{\alpha\beta}(\ {}^L\mathbf{e}_\alpha f\ {}^L\mathbf{e}_\beta f - \ {}^L\mathbf{e}_\beta f\ {}^L\mathbf{e}_\alpha f)) + f - 2], $$

for scaling ${}^2\phi = -f/2$ *in* ${}^2\mathcal{W}$, ${}^3\phi = (\ln|f-2| - f)/2$ *in* ${}^3\mathcal{W}$.

Conclusion: The Ricci flow theory of almost Kähler – Finsler/ -Lagrange / -Einstein spaces can be extracted from noncommutative geometry.

page 30: Ricci Solitons & DQ

Aim: Perform DQ using N–adapted frames (for Fedosov operators), the Cartan d–connection and distortions with Neijenhuis tensor, $\to$ star product.

$$\check{\Gamma}^{\alpha'}_{\beta'\gamma'} = \check{\mathbf{e}}^{\alpha'}_{\ \alpha}\check{\mathbf{e}}^{\ \beta}_{\beta'}\check{\mathbf{e}}^{\ \gamma}_{\gamma'}\Gamma^{\alpha}_{\beta\gamma} + \check{\mathbf{e}}^{\alpha'}_{\ \alpha}\mathbf{e}_{\gamma}(\check{\mathbf{e}}^{\ \alpha}_{\beta'}),\ \check{\Gamma}' = \Gamma + \check{Z}$$

$\check{\mathbf{e}}_{\nu'} = \check{\mathbf{e}}^{\ v}_{v'}(u)\mathbf{e}_{\nu}$ $\check{\mathbf{e}}^{\nu'} = \check{\mathbf{e}}^{v'}_{\ v}(u)\mathbf{e}^{v}$, new sets $\check{\mathbf{N}} = \{\check{N}^{a'}_{j}\}$ when $\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma} = (1/4)\check{\Omega}^{\alpha}_{\ \beta\gamma}$.

"Formal power" series and Wick product

$C^{\infty}(\mathbf{V})[[\ell]]$ of "formal series" on ℓ with coefficients from $C^{\infty}(\mathbf{V})$ on a Poisson $(\mathbf{V}, \{\cdot,\cdot\})$, where the bracket $\{\cdot,\cdot\}$. Operator

$$^{1}f * {}^{2}f = \sum_{r=0}^{\infty} {}_{r}C({}^{1}f, {}^{2}f)\, \ell^{r},$$

${}_{r}C, r \geq 0$, are bilinear operators with ${}_{0}C({}^{1}f, {}^{2}f) = {}^{1}f\,{}^{2}f$ and ${}_{1}C({}^{1}f, {}^{2}f) - {}_{1}C({}^{2}f, {}^{1}f) = i\{{}^{1}f, {}^{2}f\}$; $i^{2} = -1$; an associative algebra structure on $C^{\infty}(\mathbf{V})[[\ell]]$ with a ℓ–linear and ℓ–addical continuous star product.

page 31

Local coordinates $(u, z) = (u^{\alpha}, z^{\beta})$, on $T\mathbf{V}$; elements as series

$$a(v, z) = \sum_{r \geq 0, |\{\alpha\}| \geq 0} a_{r,\{\alpha\}}(u) z^{\{\alpha\}} \ \ell^{r}, \ \ \text{is a multi–index}\{\alpha\}$$

On $T_u\mathbf{V}$, a formal Wick product with $\check{\Lambda}^{\alpha\beta} := \check{\theta}^{\alpha\beta} - i\ \check{\mathbf{g}}^{\alpha\beta}$,

$$a \circ b\,(z) := \exp\left(i\frac{\ell}{2}\ \check{\Lambda}^{\alpha\beta} \frac{\partial^2}{\partial z^{\alpha} \partial z^{\beta}_{[1]}} \right) a(z) b(z_{[1]})\ |_{z = z_{[1]}}$$

The d–connection extended on space $\check{\mathcal{W}} \otimes \check{\Lambda}$ to operator

$$\check{\mathbf{D}}\,(a \otimes \xi) := \left(\check{\mathbf{e}}_{\alpha}(a) - u^{\beta}\ \check{\Gamma}^{\gamma}_{\alpha\beta}\ {}^{z}\check{\mathbf{e}}_{\alpha}(a) \right) \otimes (\check{\mathbf{e}}^{\alpha} \wedge \xi) + a \otimes d\xi,$$

where ${}^{z}\check{\mathbf{e}}_{\alpha}$ is a similar to $\check{\mathbf{e}}_{\alpha}$ but depend on z–variables. This operator is a N–adapted $\deg_a$–graded derivation of the d–algebra $\left(\check{\mathcal{W}} \otimes \check{\Lambda}, \circ\right)$.

page 32: Fedosov N–adapted operators

Definition: The Fedosov N–adapted operators are

$$\check{\delta}(a) = \check{\mathbf{e}}^{\alpha} \wedge {}^{z}\check{\mathbf{e}}_{\alpha}(a) \text{ and } \check{\delta}^{-1}(a) = \begin{cases} \frac{i}{p+q} z^{\alpha}\, \check{\mathbf{e}}_{\alpha}(a), \text{ if } p+q>0, \\ 0, \text{ if } p=q=0, \end{cases}$$

$a \in \check{\mathcal{W}} \otimes \Lambda$ is homogeneous w.r.t. the grading $\deg_s(a) = p, \deg_a(a) = q$.

Theorem: Any d-metric/ equivalent symplectic structure, $\check{\theta}(\cdot,\cdot) := \mathbf{g}(\check{\mathbf{J}}\cdot,\cdot)$, define a flat canonical Fedosov d–connection $\check{\mathcal{D}} : -\check{\delta} + \check{\mathbf{D}} - \frac{i}{\ell} ad_{Wick}(r)$; $\check{\mathcal{D}}^2 = 0$; $\exists$ a unique element $r \in \check{\mathcal{W}} \otimes \check{\Lambda}$, $\deg_a(r) = 1$, $\check{\delta}^{-1} r = 0$, solving $\check{\delta} r = \check{\mathcal{T}} + \check{\mathcal{R}} + \check{\mathbf{D}} r - \frac{i}{\ell} r \circ r$. Recursively,

$$\begin{aligned} r^{(0)} &= r^{(1)} = 0,\ r^{(2)} = \check{\delta}^{-1}\check{\mathcal{T}},\ r^{(3)} = \check{\delta}^{-1}(\check{\mathcal{R}} + \check{\mathbf{D}} r^{(2)} - \frac{i}{\ell} r^{(2)} \circ r^{(2)}), \\ r^{(k+3)} &= \check{\delta}^{-1}(\check{\mathbf{D}} r^{(k+2)} - \frac{i}{\ell} \sum_{l=0}^{k} r^{(l+2)} \circ r^{(l+2)}), k \geq 1, \end{aligned}$$

$a^{(k)}$ is the *Deg*–homogeneous component of degree k of $a \in \check{\mathcal{W}} \otimes \check{\Lambda}$.

page 33: Main theorems for Fedosov–Ricci solitons

Analogs of torsion and curvature operators of $\check{\mathbf{D}}$ on $\check{\mathcal{W}} \otimes \check{\Lambda}$,

$$\check{\mathcal{T}} := \frac{z^{\gamma}}{2}\, \check{\theta}_{\gamma\tau}\, \check{\mathbf{T}}^{\tau}_{\alpha\beta}(u)\, \check{\mathbf{e}}^{\alpha} \wedge \check{\mathbf{e}}^{\beta}, \qquad \check{\mathcal{R}} := \frac{z^{\gamma} z^{\varphi}}{4} \check{\theta}_{\gamma\tau} \check{\mathbf{R}}^{\tau}{}_{\varphi\alpha\beta}(u)\, \check{\mathbf{e}}^{\alpha} \wedge \check{\mathbf{e}}^{\beta}$$

Properties: $\left[\check{\mathbf{D}}, \check{\delta}\right] = \frac{i}{\ell} ad_{Wick}(\check{\mathcal{T}})$ and $\check{\mathbf{D}}^2 = -\frac{i}{\ell} ad_{Wick}(\check{\mathcal{R}})$.
The bracket $[\cdot,\cdot]$ is the $\deg_a$–graded commutator of endomorphisms of $\check{\mathcal{W}} \otimes \check{\Lambda}$ and ad_{Wick} is defined via the $\deg_a$–graded commutator in $\left(\check{\mathcal{W}} \otimes \check{\Lambda}, \circ\right)$.

Theorem 1: A star–product for the almost Kähler model

of a nonholonomic Ricci soliton is defined on $C^{\infty}(\mathbf{V})[[\ell]]$ by

$${}^{1}f * {}^{2}f \doteqdot \sigma(\tau({}^{1}f)) \circ \sigma(\tau({}^{2}f)),$$

where the projection $\sigma : \check{\mathcal{W}}_{K_{\mathcal{D}}} \to C^{\infty}(\mathbf{V})[[\ell]]$ onto the part of $\deg_s$–degree zero is a bijection and the inverse map $\tau : C^{\infty}(\mathbf{V})[[\ell]] \to \check{\mathcal{W}}_{\check{\mathcal{D}}}$ can be calculated recursively w.r..t the total degree Deg,$\tau(f)^{(0)} = f$,

$$\tau(f)^{(k+1)} = \check{\delta}^{-1}\left(\check{\mathbf{D}}\tau(f)^{(k)} - \frac{i}{v}\sum_{l=0}^{k} ad_{Wick}(r^{(l+2)})(\tau(f)^{(k-l)})\right), \text{ for } k \geq 0.$$

page 34:

${}^{f}\xi$ is the Hamiltonian vector field for a function $f \in C^{\infty}(\mathbf{V})$ on $(\mathbf{V}, \check{\theta})$. Antisymmetric ${}^{-}C({}^{1}f, {}^{2}f) := \frac{1}{2}\left(C({}^{1}f, {}^{2}f) - C({}^{2}f, {}^{1}f)\right)$ of bilinear $C({}^{1}f, {}^{2}f)$.
A star–product is normalized if ${}_{1}C({}^{1}f, {}^{2}f) = \frac{i}{2}\{{}^{1}f, {}^{2}f\}$, $\{\cdot, \cdot\}$ is the Poisson bracket defined by $\check{\theta}$. For a normalized $*$, the bilinear ${}_{2}^{-}C$ is a de Rham–Chevalley 2–cocycle $\exists$ a unique closed 2–form $\check{\varkappa}$, ${}_{2}C({}^{1}f, {}^{2}f) = \frac{1}{2}\check{\varkappa}({}^{f_1}\xi, {}^{f_2}\xi) \forall {}^{1}f, {}^{2}f \in C^{\infty}(\mathbf{V})$.
Consider the class c_0 of a normalized star–product $*$ as the equivalence class $c_0(*) \doteqdot [\check{\varkappa}]$, computed as a unique 2–form,

$$\check{\varkappa} = -\frac{i}{8}\check{\mathbf{J}}_{\tau}^{\ \alpha'}\check{\mathbf{R}}^{\tau}{}_{\alpha'\alpha\beta}\,\check{\mathbf{e}}^{\alpha} \wedge \check{\mathbf{e}}^{\beta} - i\,\check{\lambda},\ \text{for}\ \check{\lambda} = d\,\check{\mu},\ \check{\mu} = \frac{1}{6}\check{\mathbf{J}}_{\tau}^{\ \alpha'}\ \check{\mathbf{T}}^{\tau}{}_{\alpha'\beta}\,\check{\mathbf{e}}^{\beta}.$$

The h- and v–projections $h\Pi = \frac{1}{2}(Id_h - iJ_h)$ and $v\Pi = \frac{1}{2}(Id_v - iJ_v)$.
The final step is to compute the closed Chern–Weyl form
$\check{\gamma} = -iTr\left[(h\Pi, v\Pi)\,\check{\mathbf{R}}\,(h\Pi, v\Pi)^{T}\right] = -iTr\left[(h\Pi, v\Pi)\,\check{\mathbf{R}}\right] = -\frac{1}{4}\check{\mathbf{J}}_{\tau}^{\ \alpha'}\check{\mathbf{R}}^{\tau}{}_{\alpha'\alpha\beta}\,\check{\mathbf{e}}^{\alpha} \wedge \check{\mathbf{e}}^{\beta}$.
The canonical class is $\check{\varepsilon} := [\check{\gamma}] \rightarrow$ proof of

Theorem 2: The zero–degree cohomology coefficient $c_0(*)$ for the almost Kähler

model of a nonholonomic Ricci soliton is $c_0(*) = -(1/2i)\,\check{\varepsilon}$.

slide 35: Conclusions & Perspectives

Key results

- (Non) Commutative Ricci flow evolution theory for almost Kähler models of Lie algebroids endowed with canonical N–connection structure
- Decoupling property of the Ricci soliton eqs for nonholonomic Lie algebroids and exact solutions in Modified Gravity
- Deformation quantization of almost Kähler geometries and physical models

Directions for future

- Supersymmetric Ricci flows, quantum groups and deformation / geometric quantization of Lie algebroids
- Noncommutative Ricci flows on Lie algebroids, Dirac operators, spectral triples, generalized symplectic structures, quantum group models
- Modified gravity theories and algebroid Ricci solitions
- Exact solutions with generalized Lie algebroid symmetries (cosmological scenarios, brane models with generalized symmetries)

T H A N K S !