The Geometric Algebra of Fierz identities

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- C. I. Lazaroiu, E. M. Babalic, I. A. Coman, *"Geometric algebra techniques in flux compactifications (I)"*, arXiv:1212.6766 [hep-th]
- C. I. Lazaroiu, E. M. Babalic, *"Geometric algebra techniques in flux compactifications (II)"* arXiv:1212.6918 [hep-th]
- C. I. Lazaroiu, E. M. Babalic, I. A. Coman, *"The geometric algebra of Fierz identities in arbitrary dimensions and signatures"*, to appear

Introduction

Geometric Algebra is an approach to the differential and spin geometry of pseudo-Riemannian manifolds (M, g) which allows for a synthetic and effective formulation of those operations on forms and spinors that can be constructed naturally by using only the differential and Riemannian structure. It has a number of advantages arising from the category-theoretical fact that it provides a **functorial realization** of the Clifford bundle of a pseudo-Riemannian manifold, thereby solving a number of issues which plague the usual approach to spin geometry.

It employs an isomorphic realization of the Clifford bundle $\mathrm{Cl}(T^*M)$ of T^*M as the Kähler-Atiyah bundle $(\wedge T^*M, \diamond)$, where $\diamond : \wedge T^*M \times \wedge T^*M \to \wedge T^*M$ is the geometric product, an associative (but non-commutative) fiberwise composition which makes the exterior bundle into a bundle of unital associative algebras.

Has natural physical interpretation through the **quantization of spin systems**, most elegantly expressed as a form of "vertical geometric quantization" of a spinning particle moving on (M, g), which shows that $\diamond$ can be viewed as a kind of star product in the sense of deformation or geometric quantization.

It allows for a *functorial* reformulation of the differential and spin geometry of pseudo-Riemannian manifolds, which is extremely effective in supergravity/string theories, especially in the presence of fluxes. Leads to deep connections with **non-commutative algebraic geometry** (the theory of **Azumaya varieties**) thereby allowing exchange of methods with that field of research.

History and outlook

     

- First inklings in Kähler's work on the Kähler-Dirac equation (1960's); some ideas used by Atiyah (1970s). Precursors: Cartan and Chevalley's algebraic spinors, the Riesz-Chevalley isomorphism.
- Basic work in General Relativity by W. Graf (1978) and few others (Estabrook, Wahlquist etc.) (1990s-2000s).
- Deep connections with Kähler-Cartan theory, in particular with Kobayashi's reformulation thereof.

Notes

- Little work was done on supergravity/string theory, despite the power of this approach.
- Connection with quantization became clearer only since 2004 (P. Henselder et al.); full analysis leads to new ideas in the spin geometry of almost Hermitian manifolds (J-P. Michel et. al).
- Implications for operations on cohomology, spin structures and the characteristic classes of spinor bundles remain unexplored (implicit in ideas of J. Vanžura, A. Trautman, T. Friedrich but not worked out in GA language).

Upshot

We use Geometric Algebra techniques to re-formulate and solve hard computational problems related to supersymmetric actions and backgrounds in supergravity compactifications of String/M Theory.

- (M, g) is a (smooth, Hausdorff and paracompact) pseudo-Riemannian manifold of signature (p, q) and dimension $d = p + q$.
- $\Lambda T^*M := \oplus_{k=0}^{d} \Lambda^k T^*M$ is the exterior bundle of M (endowed with the metric induced by g). $\Omega(M) := \mathcal{C}^\infty(\Lambda T^*M)$ is the space of all (inhomogeneous) differential forms on M.
- S is a *real* pin bundle of M, defined as a bundle of simple modules over the Clifford bundle $\mathrm{Cl}(T^*M)$. It is well-known (Trautmann and Friedrich) that that S exists iff M has a $\mathrm{Clifford}^c$ structure, in which case it is the $\mathrm{Clifford}^c$ spinor bundle of M.
- $\iota : \Omega^*(M) \to \Omega^*(M)$ is the *left interior product* (a.k.a. *left generalized contraction*) operator, defined as the adjoint of the wedge product: $\langle \iota_\omega \eta, \rho \rangle = \langle \omega, \eta \wedge \rho \rangle$ for all $\omega, \eta, \rho \in \Omega(M)$.

The Kähler-Atiyah bundle

Definition The geometric product of (M, g) is the unique associative and unital bundle morphism $\diamond : \wedge T^*M \otimes \wedge T^*M \to \wedge T^*M$ which satisfies the *Riesz-Chevalley formulas*:

$$\begin{aligned} X \diamond \omega &= X \wedge_x \omega + \iota_X \omega \\ \omega \diamond X &= (-1)^k (X \wedge \omega - \iota_X \omega) \end{aligned}$$

for all $X \in \Gamma(M, T^*M)$ and $\omega \in \Omega^k(M)$. When endowed with this composition, the bundle of algebras $(\wedge T^*M, \diamond)$ is called the Kähler-Atiyah bundle of (M, g); it is isomorphic (as a bundle of unital associative algebras) with the Clifford bundle $\mathrm{Cl}(T^*M)$.

Note The pin bundle S can be viewed as a bundle of modules over the Kähler-Atiyah bundle. We let $\gamma : \wedge T^*M \to \mathrm{End}(S)$ be the (unital) morphism of bundles of associative algebras defining this module structure; it is fiberwise equivalent with a representation of the Clifford algebra $\mathrm{Cl}(T^*_x M)$ on the fiber S_x.

There exists a semiclassical expansion of the geometric product:

$$\diamond = \sum_{k=0}^{[\frac{d}{2}]} (-1)^k \, \triangle_{2k} + \sum_{k=0}^{[\frac{d-1}{2}]} (-1)^{k+1} \, \triangle_{2k+1} \circ (\pi \otimes \mathrm{id}_{\wedge T^*M}) \quad , \tag{1}$$

where $\Delta_k : \wedge T^*M \otimes \wedge T^*M \to \wedge T^*M$ $(k = 0, \ldots, d)$ are the generalized products, defined inductively through:

$$\omega \Delta_0 \eta = \omega \wedge \eta \quad , \quad \omega \Delta_{k+1} \eta = \frac{1}{k+1} g_{mn} (\iota_{e^m} \omega) \Delta_k (\iota_{e^n} \eta) \quad .$$

Notes:

- Δ_k is homogeneous of degree $-2k$
- Δ_k are the homogeneous components of $\diamond$

Theorem Any natural and smooth multilinear *algebraic* fiberwise operation on $\wedge T^*M$ which can be constructed using only the metric and differential structure of (M, g) can be expressed as a combination of geometric products and the operation of taking rank components.

Notes:

- *Natural* means functorial while *smooth* means *smooth functor* in the sense of Serge Lang.
- A precursor of this theorem was given by Leo Dorst.

Vertical Quantization

Theorem When (M, g) admits a compatible almost complex structure J, the operation $\diamond$ can be identified with the star product of "vertical" Weyl quantization of a certain even symplectic supermanifold associated with (M, g) (with polarization induced by J).

Notes

- In the flat case, this form of "vertical quantization" is well-known (Berezin & Marinov (1967)). In the curved (esp. compact) case, rigorous results are quite recent (J-P. Michel).
- The physical interpretation is given by the spinning particle moving on (M, g) in the presence of fluxes. The most general coupling can be written down by generalizing work of Van Holten and Riedtijk and makes connection with the theory of generalized Dirac operators as developed by E. Getzler and in the book of Berline, Getzler and Vergne.
- There exist implications for index theorems (with or without fluxes) and characteristic classes.
- Conditions for (M, g) to admit a compatible almost complex structure are non-trivial in higher dimensions. For $d = 8$ (of interest for M/F -theory compactifications, higher Donaldson/Seiberg-Witten theory etc.) those conditions were worked out quite recently by J. Vanžura et al.

Upshot The expansion of the geometric product into generalized products can be seen as the semiclassical expansion of a star product (upon replacing the metric with $\frac{g}{\hbar}$). The complexity of all natural operations on differential forms (induced by various ways of combining the wedge product with contractions of indices) is the well-known complexity characteristic of the semiclassical expansion of quantum operations. Just as Heisenberg simplified the theory of quantum observables and amplitudes by introducting the operator formalism, one can simplify the analysis of operations on differential forms and spinors by using the quantum language provided by the geometric product. In particular, this is a baby version of quantum geometry — the full string variant of which would correspond to performing the corresponding analysis on the loop space of (M, g).

Computational aspects Since the definition of generalized products is recursive, Geometric Algebra is highly amenable to implementation in symbolic domain systems such as `Mathematica` (`Ricci`, `GrassmannAlgebra`, `MathTensor`), `Maple` (`Clifford`, `GfG/TNB`), `Cadabra`, `Singular/Plural`, etc. This allows us to re-formulate succintly and promises to almost fully automate hard computations which used to be the bane of supergravity. As one application, we used this approach in the study of certain flux compactifications of M-theory, which were never studied in full generality before. The domain of applications is extremely wide, comprising the whole subject of "spin geometry" as defined by Lawson and Michelson.

Implementation. I wrote procedures (being generalized by Ioana) to implement this approach within:

- `Ricci` (a `Mathematica` package for tensor computations)
- `GrassmannAlgebra` (a `Mathematica` package for computation with multilinear forms)
- `Cadabra` (a specialized symbolic computation system written in $\mathtt{C}^{++}$)

Generalized Killing spinors. I developed a mathematical theory of generalized Killing spinors, connecting it to a notion of generalized Killing forms. This is quite technical and explained in some detail in our papers, but not directly relevant to this talk. There are deep connections with Kähler-Cartan theory, especially in its formulation via jet bundles (Kobayashi). Also deep connections with non-commutative algebraic geometry (the theory of Azumaya varieties). All this promises to change the point of view on supergravity and string theory compactifications, especially in the singular context.

pinor bundle: $\mathbb{R}$-vector bundle endowed with a morphism of algebras $\gamma : (\Omega(M), \diamond) \to (\mathrm{End}(S), \circ)$ which makes S into a bundle of modules over the the Kähler-Atiyah bundle of (M, g).

pin bundle: A spinor bundle for which S is a bundle of *simple* modules over the Kähler-Atiyah bundle.

spin(or) bundle: As above, but replace the Kähler-Atiyah bundle with its even rank sub-bundle.

Notes:

- This is more general (and in some ways better) than Cartan's approach via $\mathrm{Spin}, \mathrm{Pin}, \mathrm{Pin}^c$ structures etc. It is also *functorial*.
- Any pinor bundle is naturally a spinor bundle.
- The (s)pin bundle case corresponds to spin=1/2.
- There exists a notion of spin projector etc. which can be defined in some cases.
- Topological and representation-theoretic subtleties will largely be suppressed below.

	$\nu \diamond \nu = +1$	$\nu \diamond \nu = -1$
ν is central	$\mathbf{1}(\mathbb{R}), \mathbf{5}(\mathbb{H})$	$\mathbf{3}(\mathbb{C}), \mathbf{7}(\mathbb{C})$
ν is not central	$\mathbf{0}(\mathbb{R}), \mathbf{4}(\mathbb{H})$	$\mathbf{2}(\mathbb{R}), \mathbf{6}(\mathbb{H})$

Table: Properties of the (real) volume form ν according to the mod 8 reduction of $p-q$. At the intersection of each row and column, we indicate the values of $p-q \pmod 8$ for which the volume form ν has the corresponding properties. In parentheses, we also indicate the Schur algebra $\mathbb{S}$ for that value of $p-q \pmod 8$. The real Clifford algebra $\mathrm{Cl}(p,q)$ is non-simple iff. $p-q \equiv_8 1,5$, which corresponds to the upper left cell of the table and is also indicated through the magenta color of that table cell. In the non-simple cases, there are two choices for γ, which are distinguished by the signature $\epsilon_\gamma = \pm 1$; these are also the only cases when γ fails to be injective. Notice that ν is central iff. d is odd. The green color indicates those values of $p-q \pmod 8$ for which a spin endomorphism can be defined (see the main text).

	injective	non-injective
surjective	$\mathbf{0}(\mathbb{R}), \mathbf{2}(\mathbb{R})$	$\mathbf{1}(\mathbb{R})$
non-surjective	$\mathbf{3}(\mathbb{C}), \mathbf{7}(\mathbb{C}), \mathbf{4}(\mathbb{H}), \mathbf{6}(\mathbb{H})$	$\mathbf{5}(\mathbb{H})$

Table: Fiberwise character of real pin representations γ. At the intersection of each row and column, we indicate the values of $p-q \pmod 8$ for which the map induced by γ on each fiber of the Kähler-Atiyah algebra has the corresponding properties. In parentheses, we also indicate the Schur algebra $\mathbb{S}$ of γ for that value of $p-q \pmod 8$. Note that γ is fiberwise surjective exactly for the normal case, i.e. when the Schur algebra is isomorphic with $\mathbb{R}$. Also notice that γ fails to be fiberwise injective precisely in the non-simple case $p-q \equiv_8 1,5$, which we indicate through the magenta colouring of the corresponding table cells.

Type	$p-q$ mod 8	$\mathbb{S}$
normal	$\mathbf{0}, \mathbf{1}, \mathbf{2}$	$\mathbb{R}$
almost complex	$\mathbf{3}, \mathbf{7}$	$\mathbb{C}$
quaternionic	$\mathbf{4}, \mathbf{5}, \mathbf{6}$	$\mathbb{H}$

Table: Type of the pin bundle of (M,g) according to the mod 8 reduction of $p-q$. The pin bundle S is called *normal*, *almost complex* or *quaternionic* depending on whether its Schur algebra is isomorphic with $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The non-simple sub-cases are indicated in magenta, while those cases when a spin operator can be defined are indicated in green.

$\mathbb{S}$	$p-q$ mod 8	$\wedge T^*_x M \approx \mathrm{Cl}(p,q)$	Δ	N	Number of choices for γ	$\gamma_x(\wedge T^*_x M)$	Fiberwise injectivity of γ
$\mathbb{R}$	0, 2	$\mathrm{Mat}(\Delta,\mathbb{R})$	$2^{[\frac{d}{2}]}=2^{\frac{d}{2}}$	$2^{[\frac{d}{2}]}$	1	$\mathrm{Mat}(\Delta,\mathbb{R})$	injective
$\mathbb{H}$	4, 6	$\mathrm{Mat}(\Delta,\mathbb{H})$	$2^{[\frac{d}{2}]-1}=2^{\frac{d}{2}-1}$	$2^{[\frac{d}{2}]+1}$	1	$\mathrm{Mat}(\Delta,\mathbb{H})$	injective
$\mathbb{C}$	3, 7	$\mathrm{Mat}(\Delta,\mathbb{C})$	$2^{[\frac{d}{2}]}=2^{\frac{d-1}{2}}$	$2^{[\frac{d}{2}]+1}$	1	$\mathrm{Mat}(\Delta,\mathbb{C})$	injective
$\mathbb{H}$	5	$\mathrm{Mat}(\Delta,\mathbb{H})^{\oplus 2}$	$2^{[\frac{d}{2}]-1}=2^{\frac{d-3}{2}}$	$2^{[\frac{d}{2}]+1}$	2 ($\epsilon_\gamma=\pm 1$)	$\mathrm{Mat}(\Delta,\mathbb{H})$	non-injective
$\mathbb{R}$	1	$\mathrm{Mat}(\Delta,\mathbb{R})^{\oplus 2}$	$2^{[\frac{d}{2}]}=2^{\frac{d-1}{2}}$	$2^{[\frac{d}{2}]}$	2 ($\epsilon_\gamma=\pm 1$)	$\mathrm{Mat}(\Delta,\mathbb{R})$	non-injective

Table: Summary of pin bundle types. The non-negative integer $N \stackrel{\text{def.}}{=} \mathrm{rk}_{\mathbb{R}} S$ is the real rank of S while $\Delta \stackrel{\text{def.}}{=} \mathrm{rk}_{\Sigma_\gamma} S$ is the Schur rank of S. The non-simple cases are indicated by the magenta shading of the corresponding table cells.

$p-q$ mod 8	$\mathrm{Cl}(p,q)$	γ is injective	ϵ_γ	$\mathcal{R}$ (real spinors)	$\nu\diamond\nu$	ν is central
0	simple	Yes	N/A	$\gamma(\nu)$ (Majorana-Weyl)	$+1$	No
1	non-simple	No	± 1	N/A	$+1$	Yes
2	simple	Yes	N/A	N/A	-1	No

Table: Summary of subcases of the normal case.

$p-q$ mod 8	$\mathrm{Cl}(p,q)$	γ is injective	ϵ_γ	D^2	$\mathcal{R}$ (real spinors)	$\nu\diamond\nu$	ν is central
3	simple	Yes	N/A	$-\mathrm{id}_S$	N/A	-1	Yes
7	simple	Yes	N/A	$+\mathrm{id}_S$	D (Majorana)	-1	Yes

Table: Summary of subcases of the almost complex case. In this case, $\gamma(\nu)$ defines a complex structure J on S and we have $\mathrm{im}\gamma = \mathrm{End}_{\mathbb{C}}(S)$. We also have an endomorphism D of S which anticommutes with J (thus giving a complex conjugation on S, when the latter is viewed as a complex vector bundle) and satisfies $[D,\gamma^m]_{+,\circ}=0$. The two subcases $p-q\equiv_8 3$ and $p-q\equiv_8 7$ are distinguished by whether D^2 equals $-\mathrm{id}_S$ or $+\mathrm{id}_S$. In both cases, γ can be viewed as an isomorphism of bundles of $\mathbb{R}$-algebras from the Kähler-Atiyah bundle $(\wedge T^*M,\diamond)$ to $(\mathrm{End}_{\mathbb{C}}(S),\circ)$, while its complexification $\gamma_{\mathbb{C}}$ gives an isomorphism of bundles of $\mathbb{C}$-algebras from the complexified Kähler-Atiyah bundle $(\wedge T^*_{\mathbb{C}}M,\diamond)$ to $(\mathrm{End}_{\mathbb{C}}(S),\circ)$. When $p-q\equiv_8 7$, D is a real structure which can be used to identify S with the complexification $(S_+)_{\mathbb{C}} \stackrel{\text{def.}}{=} S_+\otimes\mathcal{O}_{\mathbb{C}}$ of the real bundle $S_+\subset S$ of *Majorana spinors*. When $p-q\equiv_8 3$, D is a second complex structure on S, which anticommutes with the complex structure $J=\gamma(\nu)$. In that case, the operators J, D and $J\circ D$ define a global quaternionic structure on S — which, however, is not compatible with γ since D *anti*commutes with γ^m.

$p-q$ mod 8	$\mathrm{Cl}(p,q)$	γ is injective	ϵ_γ	$\mathcal{R}$ (real spinors)	$\nu\diamond\nu$	ν is central
4	simple	Yes	N/A	$\gamma(\nu)$ (sympl. Majorana-Weyl)	$+1$	No
5	non-simple	No	± 1	N/A	$+1$	Yes
6	simple	Yes	N/A	$\gamma(\nu)\circ J$ (sympl. Majorana)	-1	No

Table: Summary of subcases of the quaternionic case. J denotes any of the complex structures induced on S by the quaternionic structure.

$\mathbb{S}$	$p-q$ mod 8	$\mathrm{Cl}(p,q)$	$\mathcal{R}$	Terminology for real spinors
$\mathbb{R}$	0	simple	$\gamma(\nu)$	Majorana-Weyl
$\mathbb{C}$	7	simple	D	Majorana
$\mathbb{H}$	4	simple	$\gamma(\nu)$	symplectic Majorana-Weyl
$\mathbb{H}$	6	simple	$\gamma(\nu)\circ J$	symplectic Majorana

Table: The product structure $\mathcal{R}$ used in the construction of the spin projectors $\mathcal{P}^{\mathcal{R}}_{\pm} \stackrel{\text{def.}}{=} \frac{1}{2}(1\pm\mathcal{R})$ for those cases when they can be defined and the corresponding terminology for real spinors. When $p-q\equiv_8 6$, the endomorphism $J\in\Gamma(M,\mathrm{End}(S))$ appearing in the expression for $\mathcal{R}$ is any of the complex structures associated with the quaternionic structure of S. Notice that $\mathrm{Cl}(p,q)$ is always simple as an $\mathbb{R}$-algebra (and hence γ is fiberwise injective) in those cases when spin projectors can be defined.

d	d mod 8	$\mathbb{S}$	Δ	N	$\mathrm{Cl}(p,q)$	Irrep. image	No. of $\mathbb{R}$−irreps.	Injective ?	Chirality operator $\mathcal{R}$	$\gamma^{(d+1)}$	Name of pinors (spinors)
1	1	$\mathbb{R}$	1	1	$\mathrm{Mat}(1,\mathbb{R})^{\oplus 2}$	$\mathrm{Mat}(1,\mathbb{R})$	2	no	N/A	$\pm\mathbf{1}$	M
2	2	$\mathbb{R}$	2	2	$\mathrm{Mat}(2,\mathbb{R})$	$\mathrm{Mat}(2,\mathbb{R})$	1	yes	N/A	$\gamma^{(3)}$	M
3	3	$\mathbb{C}$	2	4	$\mathrm{Mat}(2,\mathbb{C})$	$\mathrm{Mat}(2,\mathbb{C})$	1	yes	N/A	$\pm J$	M
4	4	$\mathbb{H}$	2	8	$\mathrm{Mat}(2,\mathbb{H})$	$\mathrm{Mat}(2,\mathbb{H})$	1	yes	$\gamma(\nu)$	$\gamma^{(5)}$	SM (SMW)
5	5	$\mathbb{H}$	2	8	$\mathrm{Mat}(2,\mathbb{H})^{\oplus 2}$	$\mathrm{Mat}(2,\mathbb{H})$	2	no	N/A	$\pm\mathbf{1}$	SM
6	6	$\mathbb{H}$	4	16	$\mathrm{Mat}(4,\mathbb{H})$	$\mathrm{Mat}(4,\mathbb{H})$	1	yes	$\gamma(\nu)\circ J$	$\gamma^{(7)}$	DM (M)
7	7	$\mathbb{C}$	8	16	$\mathrm{Mat}(8,\mathbb{C})$	$\mathrm{Mat}(8,\mathbb{C})$	1	yes	D	$\pm J$	DM (M)
8	0	$\mathbb{R}$	16	16	$\mathrm{Mat}(16,\mathbb{R})$	$\mathrm{Mat}(16,\mathbb{R})$	1	yes	$\gamma(\nu)$	$\gamma^{(9)}$	M (MW)
9	1	$\mathbb{R}$	16	16	$\mathrm{Mat}(16,\mathbb{R})^{\oplus 2}$	$\mathrm{Mat}(16,\mathbb{R})$	2	no	N/A	$\pm\mathbf{1}$	M
10	2	$\mathbb{R}$	32	32	$\mathrm{Mat}(32,\mathbb{R})$	$\mathrm{Mat}(32,\mathbb{R})$	1	yes	N/A	$\gamma^{(11)}$	M
11	3	$\mathbb{C}$	32	64	$\mathrm{Mat}(32,\mathbb{C})$	$\mathrm{Mat}(32,\mathbb{C})$	1	yes	N/A	$\pm J$	DM
12	4	$\mathbb{H}$	32	128	$\mathrm{Mat}(32,\mathbb{H})$	$\mathrm{Mat}(32,\mathbb{H})$	1	yes	$\gamma(\nu)$	$\gamma^{(13)}$	SM (SMW)

Table: Clifford algebras, representations and character of spinors for Riemannian manifolds. In this case, one has $q=0$ and $d=p$.

d	$d-2$ mod 8	$\mathbb{S}$	Δ	N	$\mathrm{Cl}(p,q)$	Irrep. image	No. of $\mathbb{R}$−irreps.	Injectve ?	Chirality operator $\mathcal{R}$	$\gamma^{(d+1)}$	Name of pinors (spinors)
1	7	$\mathbb{C}$	1	2	$\mathrm{Mat}(1,\mathbb{C})$	$\mathrm{Mat}(1,\mathbb{C})$	1	yes	D	$\pm J$	DM (M)
2	0	$\mathbb{R}$	2	2	$\mathrm{Mat}(2,\mathbb{R})$	$\mathrm{Mat}(2,\mathbb{R})$	1	yes	$\gamma(\nu)$	$\gamma^{(3)}$	M (MW)
3	1	$\mathbb{R}$	2	2	$\mathrm{Mat}(2,\mathbb{R})^{\oplus 2}$	$\mathrm{Mat}(2,\mathbb{R})$	2	no	N/A	$\pm\mathbf{1}$	M
4	2	$\mathbb{R}$	4	4	$\mathrm{Mat}(4,\mathbb{R})$	$\mathrm{Mat}(4,\mathbb{R})$	1	yes	N/A	$\gamma^{(5)}$	M
5	3	$\mathbb{C}$	4	8	$\mathrm{Mat}(4,\mathbb{C})$	$\mathrm{Mat}(4,\mathbb{C})$	1	yes	N/A	$\pm J$	SM
6	4	$\mathbb{H}$	4	16	$\mathrm{Mat}(4,\mathbb{H})$	$\mathrm{Mat}(4,\mathbb{H})$	1	yes	$\gamma(\nu)\circ J$	$\gamma^{(7)}$	SM (SMW)
7	5	$\mathbb{H}$	4	16	$\mathrm{Mat}(4,\mathbb{H})^{\oplus 2}$	$\mathrm{Mat}(4,\mathbb{H})$	2	no	N/A	$\pm\mathbf{1}$	SM
8	6	$\mathbb{H}$	8	32	$\mathrm{Mat}(8,\mathbb{H})$	$\mathrm{Mat}(8,\mathbb{H})$	1	yes	$\gamma(\nu)\circ J$	$\gamma^{(9)}$	DM (W)
9	7	$\mathbb{C}$	16	32	$\mathrm{Mat}(16,\mathbb{C})$	$\mathrm{Mat}(16,\mathbb{C})$	1	yes	D	$\pm J$	DM (M)
10	0	$\mathbb{R}$	32	32	$\mathrm{Mat}(32,\mathbb{R})$	$\mathrm{Mat}(32,\mathbb{R})$	1	yes	$\gamma(\nu)$	$\gamma^{(11)}$	M (MW)
11	1	$\mathbb{R}$	32	32	$\mathrm{Mat}(32,\mathbb{R})^{\oplus 2}$	$\mathrm{Mat}(32,\mathbb{C})$	2	no	N/A	$\pm\mathbf{1}$	M
12	2	$\mathbb{R}$	64	64	$\mathrm{Mat}(64,\mathbb{R})$	$\mathrm{Mat}(64,\mathbb{R})$	1	yes	N/A	$\gamma^{(13)}$	M

Table: Clifford algebras, representations and character of spinors for Lorentzian manifolds. In this case, one has $p=d-1$, $q=1$ and $p-q=d-2$.

Given an admissible fiberwise bilinear pairing $\mathscr{B}$ on S, we define endomorphisms $E_{\xi,\xi'} \in \Gamma(M, \mathrm{End}(S)) \approx \mathrm{Hom}_{\mathcal{C}^\infty(M,\mathbb{R})}(\Gamma(M,S), \Gamma(M,S))$ through:

$$E_{\xi,\xi'}(\xi'') \stackrel{\text{def}}{=} \mathscr{B}(\xi'', \xi')\xi \quad , \quad \forall \xi, \xi' \in \Gamma(M,S) \quad .$$

One has the *algebraic Fierz identities*:

$$\boxed{E_{\xi_1,\xi_2} \circ E_{\xi_3,\xi_4} = \mathscr{B}(\xi_3,\xi_2) E_{\xi_1,\xi_4} \quad , \quad \forall \xi_1,\xi_2,\xi_3,\xi_4 \in \Gamma(M,S)} \quad , \tag{2}$$

as well as:

$$\mathrm{tr}(T \circ E_{\xi,\xi'}) = \mathscr{B}(T\xi, \xi') \quad , \quad \forall \xi, \xi' \in \Gamma(M,S) \quad . \tag{3}$$

Fierz identities for the normal case ($p - q \equiv_8 0, 1, 2$)

Proposition. We have the *completeness relation for the normal case*:

$$\boxed{T =_U \frac{N}{2^d} \sum_{A=\text{ordered}} \mathrm{tr}(\gamma_A^{-1} \circ T)\gamma_A \quad , \quad \forall T \in \Gamma(M, \mathrm{End}(S))} \quad . \tag{4}$$

Setting $T = E_{\xi,\xi'}$ in this relation gives the expansion:

$$E_{\xi,\xi'} = \frac{N}{2^d} \sum_{A=\text{ordered}} \epsilon_{\mathscr{B}}^{|A|} \mathscr{B}(\xi, \gamma^A \xi')\gamma_A \quad . \tag{5}$$

The geometric Fierz identities. The inhomogeneous differential forms:

$$\check{E}_{\xi,\xi'} \stackrel{\text{def.}}{=} \left(\gamma|_{\Omega^\gamma(M)}\right)^{-1}(E_{\xi,\xi'}) \in \Omega^\gamma(M)$$

have the expansion:

$$\boxed{\check{E}_{\xi,\xi'} = \frac{N}{2^d} \sum_{A=\text{ordered}} \epsilon_{\mathscr{B}}^{|A|} \mathscr{B}(\xi, \gamma_A \xi') e_\gamma^A \quad , \quad \forall \xi, \xi' \in \Gamma(M,S)} \quad .$$

and satisfy *the geometric Fierz identities*:

$$\boxed{\check{E}_{\xi_1,\xi_2} \diamond \check{E}_{\xi_3,\xi_4} = \mathscr{B}(\xi_3,\xi_2) \check{E}_{\xi_1,\xi_4} \quad , \quad \forall \xi_1,\xi_2,\xi_3,\xi_4 \in \Gamma(M,S)} \quad ,$$

an equality which holds in $\Omega^\gamma(M)$.

Fierz identities for the almost complex case

Proposition. We have the *partial completeness relation for the almost complex case*:

$$\boxed{\frac{2^{d+1}}{N}T = \frac{2^d}{\Delta}T =_U \sum_{A=\text{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A \quad , \quad \forall T \in \Gamma(M, \mathrm{End}_{\mathbb{C}}(S))} \quad . \tag{6}$$

and the *full completeness relation for the almost complex case*:

$$\boxed{\frac{2^{d+1}}{N}T = \frac{2^d}{\Delta}T =_U \sum_{A=\text{ordered}} \left[\mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A + \mathrm{tr}(\gamma_A^{-1}\circ D^{-1}\circ T)D\circ\gamma_A\right] \quad , \quad \forall T \in \Gamma(M, \mathrm{End}(S))} \quad . \tag{7}$$

Skipping many intermediate steps, this gives *the geometric Fierz identities for the almost complex case*:

$$\boxed{\begin{aligned} &\check{E}^{(0)}_{\xi_1,\xi_2} \diamond \check{E}^{(0)}_{\xi_3,\xi_4} + (-1)^{\frac{p-q+1}{4}} \mathbf{\Pi}(\check{E}^{(1)}_{\xi_1,\xi_2}) \diamond \check{E}^{(1)}_{\xi_3,\xi_4} = \mathscr{B}_0(\xi_3,\xi_2)\check{E}^{(0)}_{\xi_1,\xi_4} \quad , \\ &\mathbf{\Pi}(\check{E}^{(0)}_{\xi_1,\xi_2}) \diamond \check{E}^{(1)}_{\xi_3,\xi_4} + \check{E}^{(1)}_{\xi_1,\xi_2} \diamond \check{E}^{(0)}_{\xi_3,\xi_4} = \mathscr{B}_0(\xi_3,\xi_2)\check{E}^{(1)}_{\xi_1,\xi_4} \quad , \end{aligned}} \tag{8}$$

with the local expansions:

$$\boxed{\begin{aligned} &\check{E}^{(0)}_{\xi,\xi'} =_U \frac{\Delta}{2^d} \sum_{A=\text{ordered}} (-1)^{|A|}\mathscr{B}_0(\xi,\gamma^A\xi')e_A \ , \\ &\check{E}^{(1)}_{\xi,\xi'} =_U \frac{\Delta}{2^d} \sum_{A=\text{ordered}} (-1)^{\frac{p-q+1}{4}}(-1)^{|A|}\mathscr{B}_0(\xi, D\circ\gamma^A\xi')e_A \ . \end{aligned}} \tag{9}$$

Fierz Identities for the quaternionic case

Proposition. We have the *partial completeness relation for the quaternionic case*:

$$\boxed{\frac{2^{d+2}}{N}T = \frac{2^d}{\Delta}T =_U \sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A \quad , \quad \forall T\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))} \tag{10}$$

and the *full completeness relation for the quaternionic case*:

$$\boxed{\frac{2^{d+2}}{N}T = \frac{2^d}{\Delta}T =_U \sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A - \sum_{k=1}^{3}\sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ J_k\circ T)J_k\circ\gamma_A \quad ,} \tag{11}$$

for any $T\in\Gamma(M,\mathrm{End}(S))$. After some work, this gives the *geometric Fierz identities for the quaternionic case*:

$$\check{E}^{(0)}_{\xi_1,\xi_2}\diamond\check{E}^{(0)}_{\xi_3,\xi_4} - \sum_{i=1}^{3}\check{E}^{(i)}_{\xi_1,\xi_2}\diamond\check{E}^{(i)}_{\xi_3,\xi_4} = \mathscr{B}_0(\xi_3,\xi_2)\check{E}^{(0)}_{\xi_1,\xi_4} \quad ,$$

$$\check{E}^{(0)}_{\xi_1,\xi_2}\diamond\check{E}^{(i)}_{\xi_3,\xi_4} + \check{E}^{(i)}_{\xi_1,\xi_2}\diamond\check{E}^{(0)}_{\xi_3,\xi_4} + \sum_{j,k=1}^{3}\epsilon_{ijk}\check{E}^{(j)}_{\xi_1,\xi_2}\diamond\check{E}^{(k)}_{\xi_3,\xi_4} = \mathscr{B}_0(\xi_3,\xi_2)\check{E}^{(i)}_{\xi_1,\xi_4} \quad (i=1\dots 3)\ ,$$

where:

$$\check{E}^{(0)}_{\xi,\xi'} =_U \frac{\Delta}{2^d}\sum_{A=\mathrm{ordered}} \epsilon^{|A|}_{\mathscr{B}_0}\mathscr{B}_0(\xi,\gamma_A\xi')e^A_\gamma \quad , \quad \forall \xi,\xi'\in\Gamma(M,S) \quad ,$$

$$\check{E}^{(i)}_{\xi,\xi'} =_U \frac{\Delta}{2^d}\sum_{A=\mathrm{ordered}} \epsilon^{|A|}_{\mathscr{B}_0}\mathscr{B}_0(\xi,(J_i\circ\gamma_A)\xi')e^A_\gamma \quad , \quad \forall i=1\dots 3 \quad .$$

Examples and applications (with help from Ioana, Mirela)

Upshot We can put conceptual aned computational order in the theory of Fierz identities on curved space-times, in arbitrary dimensions and signatures — a notoriously messy subject which used to be the bane of supergravity/string theory.

Tests of correctness After some discussion of complexification/decomplexification, biquaternions etc. we correctly recover results which were obtained previously through much more cumbersome computations — in particular, finding full agreement with some well-known examples in each class of pin representations (normal, almost complex and quaternionic).

New applications We worked out new examples and plan to do many more. I will spare you the details, which are quite involved (see the papers).

For the future Using a generalization of our code, we hope to unifty, systematize and automate the story of Fierz identities in supegravity and string theory, as well as to analyze the many cases which have remained inaccessible due to lack of conceptual understanding as well as due to computational difficulty.

Issues that I have skipped over The (rather lengthy) proofs, some global vs. local aspects, quite a bit of the algebra/representation theory, a synthetic formulation of geometric Fierz identities which can be given using the language of superalgebras, the theory of twisted anti-selfdual inhomogeneous forms, identities for the generalized products etc. — these are treated in detail in the papers.