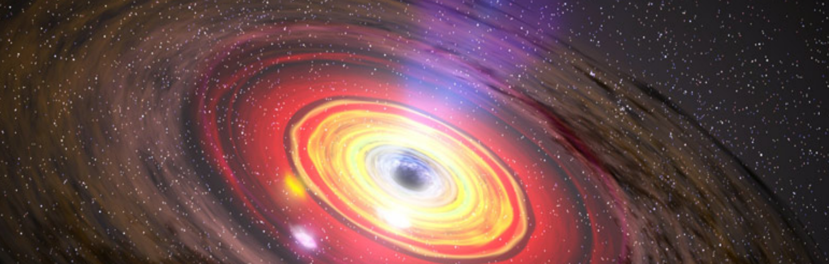
Singular General Relativity

A Geometric approach to the Singularities in General Relativity

Cristi Stoica, Universitatea Politehnică București



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It is hoped that when GR will be quantized, this will solve the singularities too, by showing probably that quantum fields prevent the occurrence of singularities. *Loop quantum cosmology* obtained significant positive results in this direction (Bojowald, 2001; Ashtekar & Singh, 2011; Vișinescu, 2009; Saha & Vișinescu, 2012)

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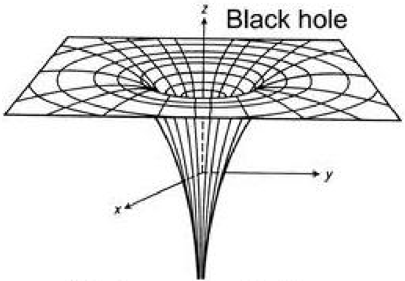
- Implications to dimensional reduction regularization in QFT and QG.

Two types of singularities



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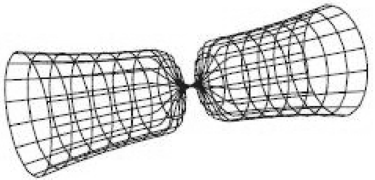

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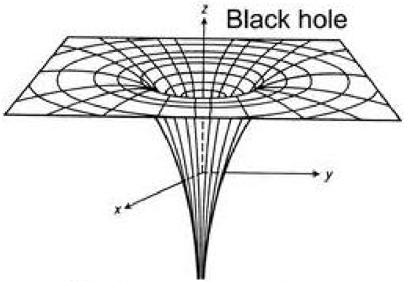

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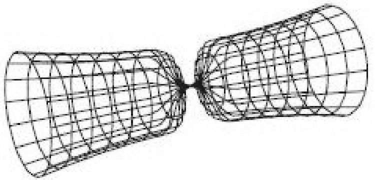

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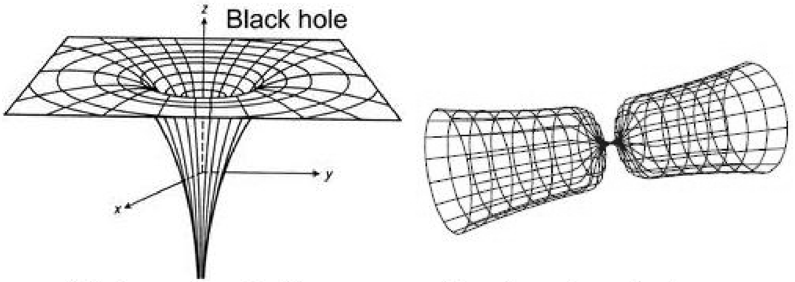

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2. **Benign singularities**: g_{ab} are smooth and finite, but $\det g \to 0$.

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1. For PDE on curved spacetimes: the covariant derivatives blow up:

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Even if g_{ab} are all finite, these equations are also in terms of g^{ab}, and $g^{ab} \to \infty$ when $\det g \to 0$.

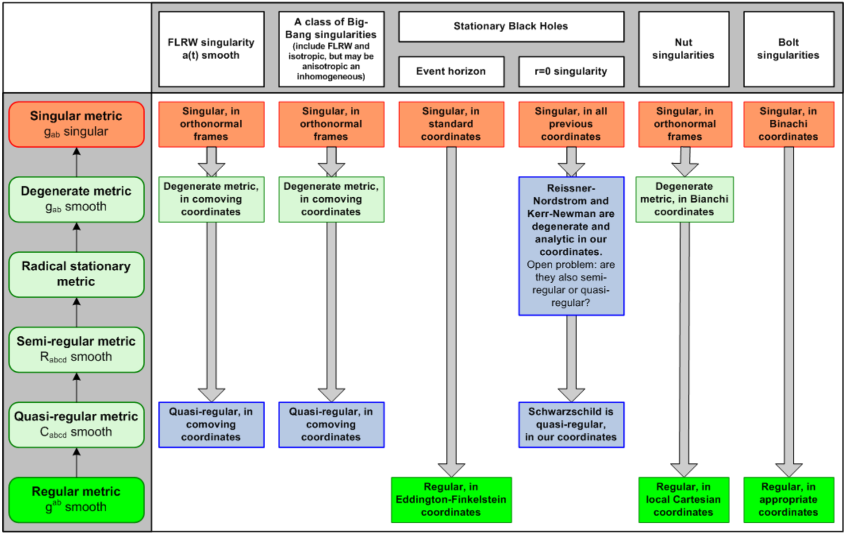
What are the non-singular objects?[1]

Some quantities which are part of the equations are indeed singular, but this is not a problem if we use instead other quantities, equivalent to them when the metric is non-degenerate.

Singular	Non-Singular	When g is...
$\Gamma^{c}{}_{ab}$ (2-nd)	Γ_{abc} (1-st)	smooth
$R^{d}{}_{abc}$	R_{abcd}	semi-regular
R_{ab}	$R_{ab}\sqrt{\lvert\det g\rvert}^{W}$, $W \leq 2$	semi-regular
R	$R\sqrt{\lvert\det g\rvert}^{W}$, $W \leq 2$	semi-regular
Ric	$\mathrm{Ric} \circ g$	quasi-regular
R	$Rg \circ g$	quasi-regular

[1](Stoica, 2011b; Stoica, 2012b)

Examples of singularities



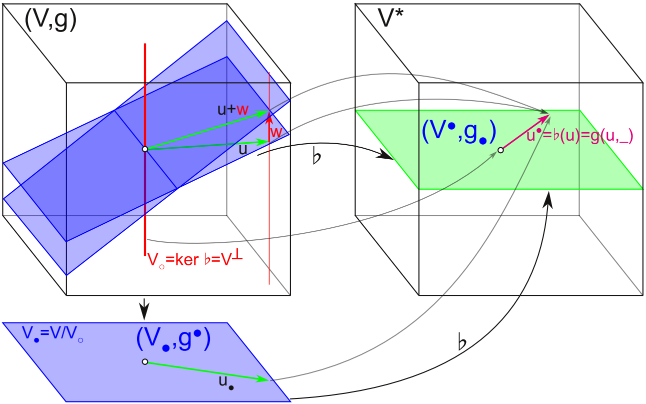
Degenerate inner product

Definition

An **inner product** on a vector space V is a symmetric bilinear form $g \in V^* \otimes V^*$. The pair (V, g) is named *inner product space*. We use alternatively the notation $\langle u, v\rangle := g(u, v)$, for $u, v \in V$. The inner product g is **degenerate** if there is a vector $v \in V$, $v \neq 0$, so that $\langle u, v\rangle = 0$ for all $u \in V$, otherwise g is *non-degenerate*. There is always a basis, named *orthonormal basis*, in which g takes a diagonal form:

$$g = \begin{pmatrix} O_r & & \\ & -I_s & \\ & & +I_t \end{pmatrix}. \tag{5}$$

where O_r is the zero operator on $\mathbb{R}^r$, and I_q, $q \in \{s, t\}$ is the identity operator in $\mathbb{R}^q$. The **signature** of g is defined as the triple (r, s, t).



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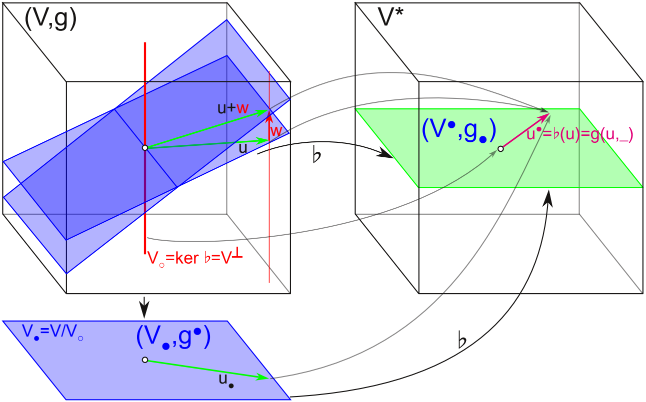
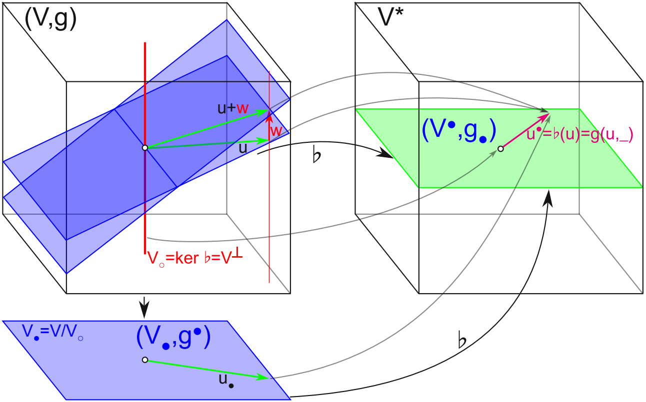


(V, g) is an inner product vector space. The morphism $\flat : V \to V^*$ is defined by $u \mapsto u^\bullet := \flat(u) = u^\flat = g(u, \_)$.



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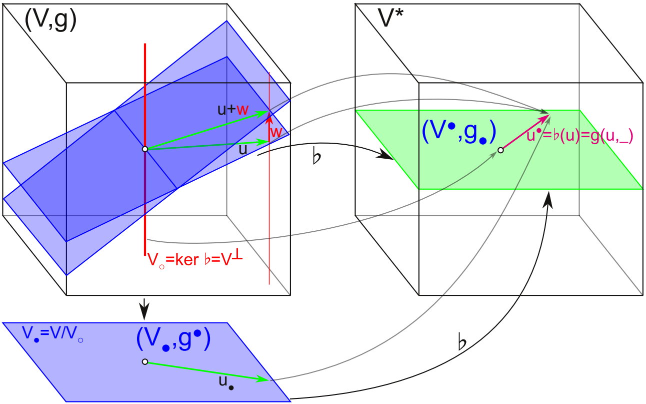

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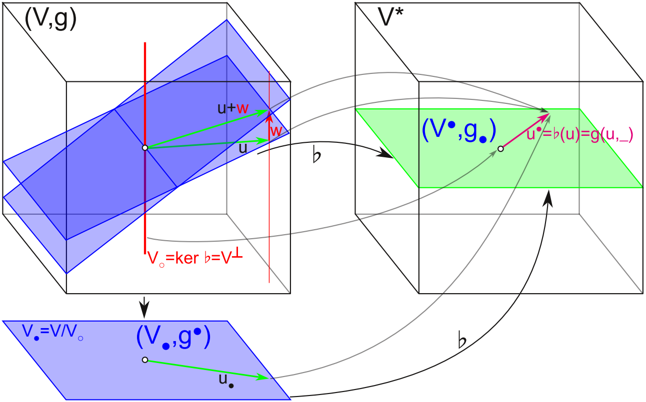

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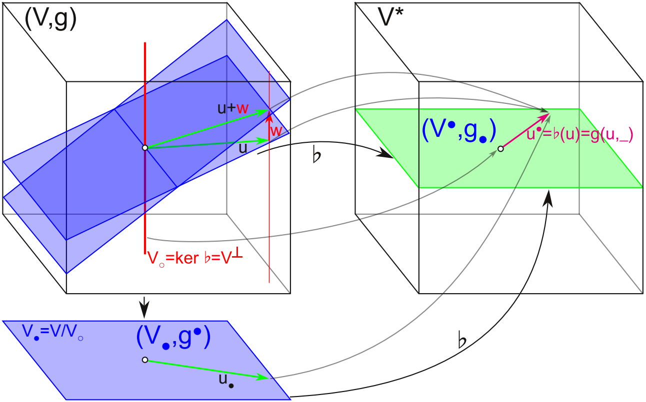

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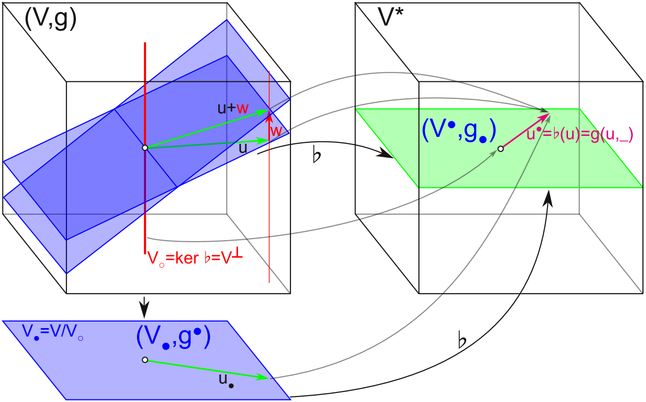

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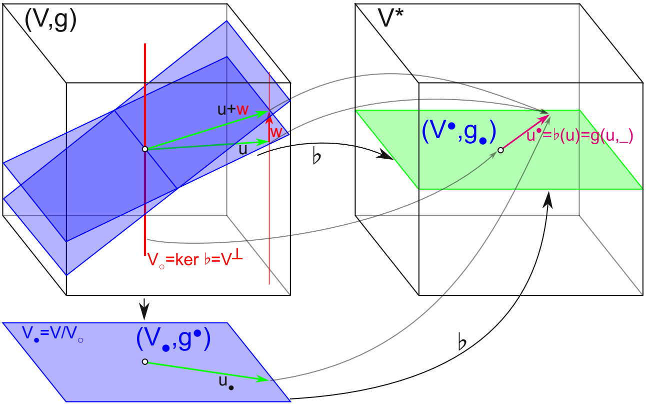

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Relations between the various spaces[2]

The relations between the radical, the radical annihilator and the factor spaces can be collected in the diagram:

$$
\begin{tikzcd}
0 \arrow[r, hook] & V_\circ \arrow[r, hook, "i_\circ"] & (V,g) \arrow[r, two heads, "\pi_\bullet"] \arrow[dr, "\flat_V"] & (V_\bullet, g^\bullet) \arrow[r, two heads] \arrow[d, bend right, "\flat"'] & 0 \\
0 & V^\circ \arrow[l, two heads] & V^* \arrow[l, two heads, "\pi^\circ"'] & (V^\bullet, g_\bullet) \arrow[l, hook', "i^\bullet"'] \arrow[u, bend right, "\sharp"'] & 0 \arrow[l, hook']
\end{tikzcd}
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where $V_\bullet = V^{\bullet *} = \dfrac{V}{V_\circ}$ and $V^\circ = {V_\circ}^* = \frac{V^*}{V^\bullet}$.

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Netric contraction between covariant indices

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$$C_{s-1\,s} := 1_{\mathcal{T}^r_{s-2}V} \otimes g_{\bullet} : \mathcal{T}^r_s V \otimes V^{\bullet} \otimes V^{\bullet} \to \mathcal{T}^r_{s-2} V, \tag{6}$$

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$$C_{kl} : V^{\otimes r} \otimes V^{*\otimes k-1} \otimes V^{\bullet} \otimes V^{*\otimes l-k-1} \otimes V^{\bullet} \otimes V^{*\otimes s-l} \to V^{\otimes r} \otimes V^{*\otimes s-2}, \tag{7}$$

by $C_{kl} := C_{s-1\,s} \circ P_{k,s-1;l,s}$, where $P_{k,s-1;l,s} : T \in \mathcal{T}^r_s V \to T \in \mathcal{T}^r_s V$ is the permutation isomorphisms which moves the k-th and l-th slots in the last two positions.

Singular semi-Riemannian manifolds

Definition

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- If g is non-degenerate, then (M, g) is a *semi-Riemannian manifold*.
- If g is positive definite, (M, g) is a *Riemannian manifold*.

Degenerate metric - algebraic properties

For the tangent bundle T_pM at a point $p \in M$, the spaces and associated metrics are defined as usual:

$$
\begin{tikzcd}
0 \arrow[r, hook] & T_{\circ p}M \arrow[r, hook, "i_\circ"] & (T_pM, g) \arrow[r, two heads, "\pi_\bullet"] \arrow[dr, "\flat_{T_pM}"] & (V_\bullet, g^\bullet) \arrow[r, two heads] \arrow[d, bend right, "\flat"'] & 0 \\
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where $T_{\bullet p}M = T^{\bullet}{}^*_pM = \dfrac{T_pM}{T_{\circ p}M}$ and $T^\circ{}_pM = (T_{\circ p}M)^* = \frac{T_p^*M}{T^\bullet{}_pM}$.

The Koszul object

The Koszul object is defined as $\mathcal{K} : \mathfrak{X}(M)^3 \to \mathbb{R}$,

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In local coordinates it is the Christoffel's symbols of the first kind:

$$\mathcal{K}_{abc} = \mathcal{K}(\partial_a, \partial_b, \partial_c) = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) = \Gamma_{abc}, \tag{9}$$

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For non-degenerate metrics, the Levi-Civita connection is obtained uniquely:

$$\nabla_X Y = \mathcal{K}(X, Y, \_)^\sharp. \tag{10}$$

The covariant derivatives[3]

The **lower covariant derivative** of a vector field Y in the direction of a vector field X:

$$(\nabla^\flat_X Y)(Z) := \mathcal{K}(X, Y, Z) \tag{11}$$

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Semi-regular manifolds. Riemann curvature tensor[4]

A **semi-regular semi-Riemannian manifold** is defined by the condition

$$\nabla_X \nabla^\flat_Y Z \in \mathcal{A}^\bullet(M). \tag{12}$$

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Examples of semi-regular semi-Riemannian manifolds[5]

- Isotropic singularities:

$$g = \Omega^2 \tilde{g}.$$

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where $k = 1$ for S^3, $k = 0$ for $\mathbb{R}^3$, and $k = -1$ for H^3.

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Einstein's equation on semi-regular spacetimes[6]

On 4D semi-regular spacetimes Einstein tensor density $G \det g$ is smooth.

At the points p where the metric is non-degenerate, the Einstein tensor can be expressed by:

$$G^{ab} \det g = g_{kl} \epsilon^{akst} \epsilon^{blpq} R_{stpq}, \tag{19}$$

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Therefore, $G_{ab} \det g$ is smooth too, and it makes sense to write a densitized version of Einstein's equation

$$G_{ab} \det g + \Lambda g_{ab} \det g = \kappa T_{ab} \det g, \tag{20}$$

where $\kappa := \dfrac{8\pi\mathcal{G}}{c^4}$, $\mathcal{G}$ and c being Newton's constant and the speed of light.

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It is not allowed to divide by $\det g$, when $\det g = 0$.

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Friedmann-Lemaître-Robertson-Walker spacetime

If S is a connected three-dimensional Riemannian manifold of constant curvature $k \in \{-1, 0, 1\}$ (*i.e.* H^3,$\mathbb{R}^3$ or S^3) and $a \in (A, B)$, $-\infty \leq A < B \leq \infty$, $a \geq 0$, then the warped product $I \times_a S$ is called a *Friedmann-Lemaître-Robertson-Walker* spacetime.

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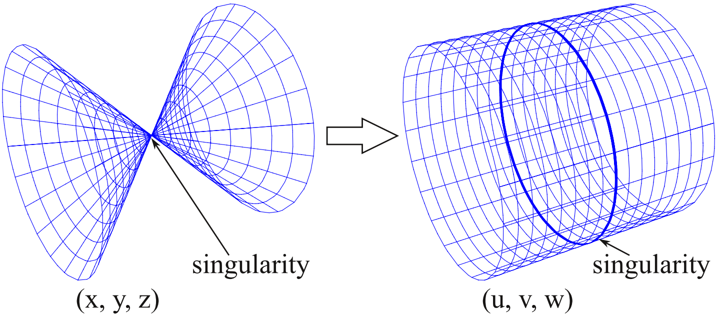
The resulting singularities are semi-regular.

Distance separation *vs.* topological separation



The old method of resolution of singularities shows how we can "untie" the singularity of a cone and obtain a cylinder.
Similarly, it is not necessary to assume that, at the Big Bang singularity, the entire space was a point, but only that the space metric was degenerate.

Friedman equations

The stress-energy tensor is

$$T^{ab} = (\rho + p)\, u^a u^b + p g^{ab}, \tag{23}$$

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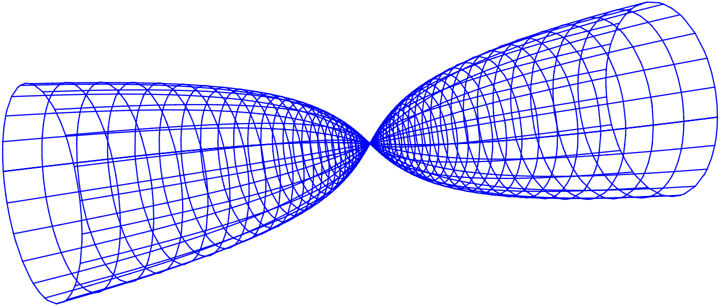
FLRW Big Bang[8]



Big Bang singularity, corresponding to $a(0) = 0$, $\dot{a}(0) > 0$.

[8](Stoica, 2011a)

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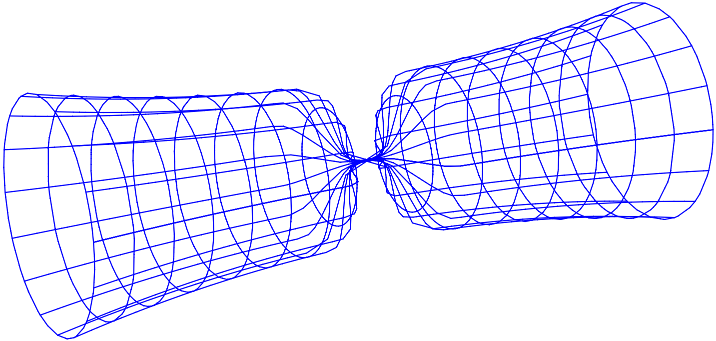

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Schwarzschild singularity is semi-regular[10]

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2, \qquad (31)$$

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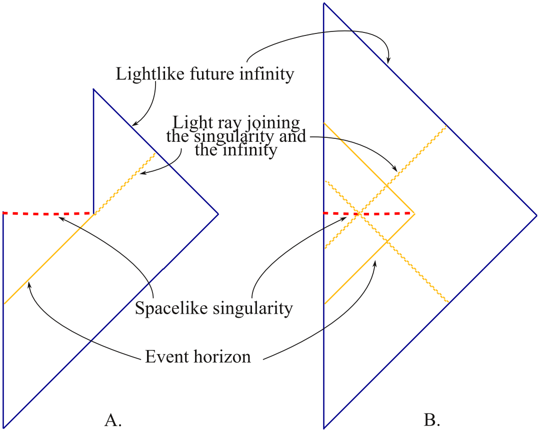
The four-metric becomes:

$$ds^2 = -\frac{4\tau^4}{2m - \tau^2} d\tau^2 + (2m - \tau^2)\tau^4 (4\xi d\tau + \tau d\xi)^2 + \tau^4 d\sigma^2 \tag{34}$$

which is analytic and semi-regular at $r = 0$.

[10] (Stoica, 2012e)

Evaporating Schwarzschild black hole and information[11]



A. Standard evaporating black hole, whose singularity destroys the information.

B. Evaporating black hole extended through the singularity preserves information.

[11] (Stoica, 2012e)

Reissner-Nordström singularity is analytic[12]

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 + r^2 d\sigma^2, \quad (35)$$

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We choose the coordinates ρ and τ, so that $\begin{cases} t &= \tau\rho^T \\ r &= \rho^S \end{cases}$

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We choose the coordinates ρ and τ, so that $\begin{cases} t &= \tau\rho^T \\ r &= \rho^S \end{cases}$

The metric has, in the new coordinates, the following form

$$\mathrm{d}s^2 = -\Delta\rho^{2T-2S-2}\left(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho\right)^2 + \frac{S^2}{\Delta}\rho^{4S-2}\mathrm{d}\rho^2 + \rho^{2S}\mathrm{d}\sigma^2, \quad (36)$$

$$\text{where} \quad \Delta := \rho^{2S} - 2m\rho^S + q^2. \quad (37)$$

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To remove the infinity of the metric at $r = 0$, take $\begin{cases} S \geq 1 \\ T \geq S + 1 \end{cases}$

which also ensure that the metric is analytic at $r = 0$.

[12] (Stoica, 2012a)

Non-singular electromagnetic field[13]

The electromagnetic potential in the coordinates (t, r, ϕ, θ) is singular at $r = 0$:

$$A = -\frac{q}{r}\mathrm{d}t, \tag{38}$$

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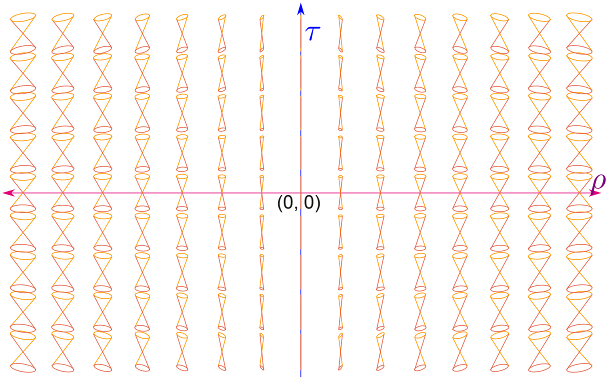
and they are analytic everywhere, including at the singularity $\rho = 0$.

[13] (Stoica, 2012a)

Null geodesics of Reissner-Nordström in our coordinates[14]

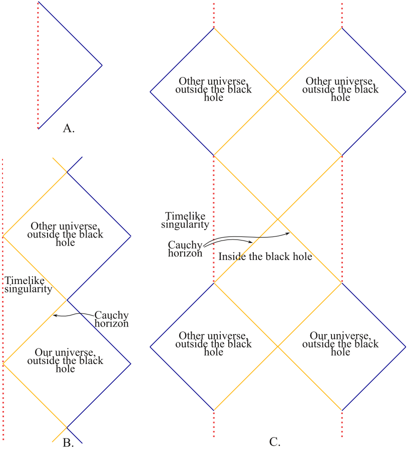
To have space+time foliation given by the coordinate, must have $T \geq 3S$.



As one approaches the singularity on the axis $\rho = 0$, the lightcones become more and more degenerate along that axis (for $T \geq 3S$ and even S).

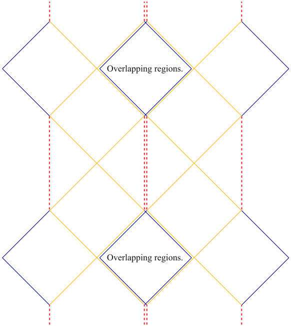
[14](Stoica, 2012a)

Penrose diagrams for the Reissner-Nordström black holes



Reissner-Nordström black holes. A. Naked solutions ($q^2 > m^2$). B. Extremal solution ($q^2 = m^2$). C. Solutions with $q^2 < m^2$.

Penrose diagram for our extension of the non-extremal Reissner-Nordström black hole [15]

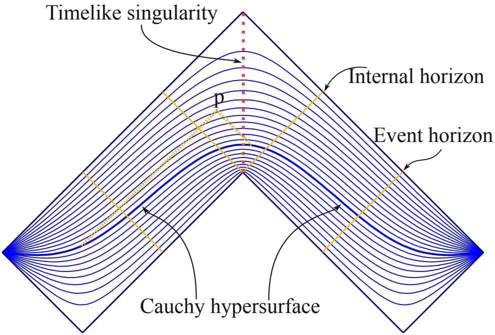


The Penrose-Carter diagram our extension of for the non-extremal Reissner-Nordström black hole with $q^2 < m^2$, analytically extended beyond the singularity. When represented in plane, it repeats periodically along both the vertical and the horizontal directions, and it has overlaps. In the diagram, there is an intentional small shift between the two copies, to make the overlapping visible.

[15] (Stoica, 2012a; Stoica, 2012f)

Space-like foliation of the Reissner-Nordström solution[16]

We can foliate the Reissner-Nordström using Cauchy hypersurfaces, if we remove the regions beyond the Cauchy horizons:



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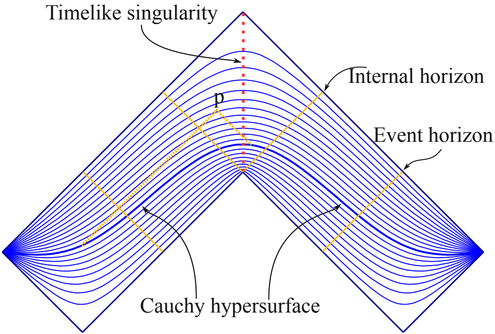

Implications: we can vary m, q, a and obtain general singularities which preserve information.

[16] (Stoica, 2012a; Stoica, 2012f)

The Ricci decomposition

The Riemann curvature tensor can be decomposed algebraically as

$$R_{abcd} = S_{abcd} + E_{abcd} + C_{abcd} \tag{41}$$

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where

$$S_{abcd} = \frac{1}{n(n-1)} R (g \circ g)_{abcd} \tag{42}$$

$$E_{abcd} = \frac{1}{n-2} (S \circ g)_{abcd} \tag{43}$$

$$S_{ab} := R_{ab} - \frac{1}{n} R g_{ab} \tag{44}$$

$$(h \circ k)_{abcd} := h_{ac} k_{bd} - h_{ad} k_{bc} + h_{bd} k_{ac} - h_{bc} k_{ad} \tag{45}$$

The expanded Einstein equation[17]

In dimension $n = 4$ we introduce the *expanded Einstein equation*

$$(G \circ g)_{abcd} + \Lambda(g \circ g)_{abcd} = \kappa(T \circ g)_{abcd} \tag{46}$$

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or, equivalently,

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It is equivalent to Einstein's equation if the metric is non-degenerate.

[17] (Stoica, 2012b)

Examples of quasi-regular singularities[18]

- Isotropic singularities.

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- Degenerate warped products $B \times_f F$ with $\dim B = 1$ and $\dim F = 3$.
- In particular, FLRW singularities.
- Schwarzschild singularities.
- The question whether the Reissner-Nordström and Kerr-Newman singularities are semi-regular, or quasi-regular, is still open.

[18] (Stoica, 2012b)

The Weyl tensor at quasi-regular singularities[19]

The *Weyl curvature tensor*:

$$C_{abcd} = R_{abcd} - S_{abcd} - E_{abcd}. \tag{48}$$

[19](Stoica, 2012c)

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For example, the spacetime proposed In (Stoica, 2012c), which is not necessarily homogeneous or isotropic, with the metric

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\mathrm{d}\sigma_t^2, \tag{49}$$

is quasi-regular, satisfying therefore the Weyl curvature hypothesis.

[19] (Stoica, 2012c)

Hints of dimensional reduction in QFT and QG

- The scattering amplitudes in QCD (Lipatov, 1988; Lipatov, 1989; Lipatov, 1991).
- High energy Regge regime (Verlinde & Verlinde, 1993; Aref'eva, 1994).
- Fractal universe (Calcagni, 2010b; Calcagni, 2010a), based on a Lebesgue-Stieltjes measure or a fractional measure (Calcagni, 2011), fractional calculus, and fractional action principles (El-Nabulsi, 2005; El-Nabulsi & Torres, 2008; Udriște & Opriș, 2008).
- Topological dimensional reduction (Shirkov, 2010; Fiziev & Shirkov, 2011; Fiziev, 2010; Fiziev & Shirkov, 2012; Shirkov, 2011).
- Vanishing Dimensions at LHC (Anchordoqui *et al.*, 2012).
- Dimensional reduction in Quantum Gravity (Carlip, 1995; Carlip *et al.*, 2009; Carlip, 2010).
- Asymptotic safety (Weinberg, 1979).
- Causal dynamical triangulations (Ambjørn *et al.*, 2000).
- Hořava-Lifschitz gravity (Hořava, 2009).

Is dimensional reduction due to the benign singularities?[20]

- Geometric, or **metric reduction**:

$$\dim T_{p\bullet}M = \dim T_p{}^{\bullet}M = \operatorname{rank} g_p. \tag{50}$$

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- Weyl tensor $C_{abcd} \to 0$ as approaching a quasi-regular singularity. This implies that the **local degrees of freedom vanish**, *i.e.* the gravitational waves for GR and the gravitons for QG (Carlip, 1995).

[20] (Stoica, 2012d)

Is dimensional reduction due to the benign singularities?[21]

[21] (Stoica, 2012d)

Is dimensional reduction due to the benign singularities?[21]

- A **charged particle** as a Reissner-Nordström black hole has dim $= 2$:

$$ds^2 = -\Delta\rho^{2T-2S-2}\left(\rho \mathrm{d}\tau + T\tau \mathrm{d}\rho\right)^2 + \frac{S^2}{\Delta}\rho^{4S-2}\mathrm{d}\rho^2 + \rho^{2S}\mathrm{d}\sigma^2. \quad (51)$$

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- To admit space+time foliation in these coordinates, we should take $T \geq 3S$. Is this anisotropy connected to **Hořava-Lifschitz** gravity?

[21] (Stoica, 2012d)

Is dimensional reduction due to the benign singularities?[22]

In the **fractal universe approach** (Calcagni, 2010b; Calcagni, 2010a; Calcagni, 2011), to resolve the problems of non-renormalizability, it was postulated that the measure in

$$S = \int_{\mathcal{M}} \mathrm{d}\varrho(x)\, \mathcal{L} \tag{52}$$

has the form

$$\mathrm{d}\varrho(x) = \prod_{\mu=0}^{D-1} f_{(\mu)}(x)\, \mathrm{d}x^{\mu}, \tag{53}$$

where some of the functions $f_{(\mu)}(x)$ vanish at low scales.
In **Singular General Relativity**, the measure postulated by Calcagni is obtained naturally, since

$$\mathrm{d}\varrho(x) = \sqrt{-\det g}\,\mathrm{d}x^{D}. \tag{54}$$

If the metric is diagonal in the coordinates (x^{μ}), then we can take

$$f_{(\mu)}(x) = \sqrt{|g_{\mu\mu}(x)|}. \tag{55}$$

[22] (Stoica, 2012d)

Quantum gravity from dimensional reduction at singularities[23]

To make GR renormalizable, some authors proposed various modifications, entailing apparently distinct kinds of dimensional reduction.

[23] (Stoica, 2012d)

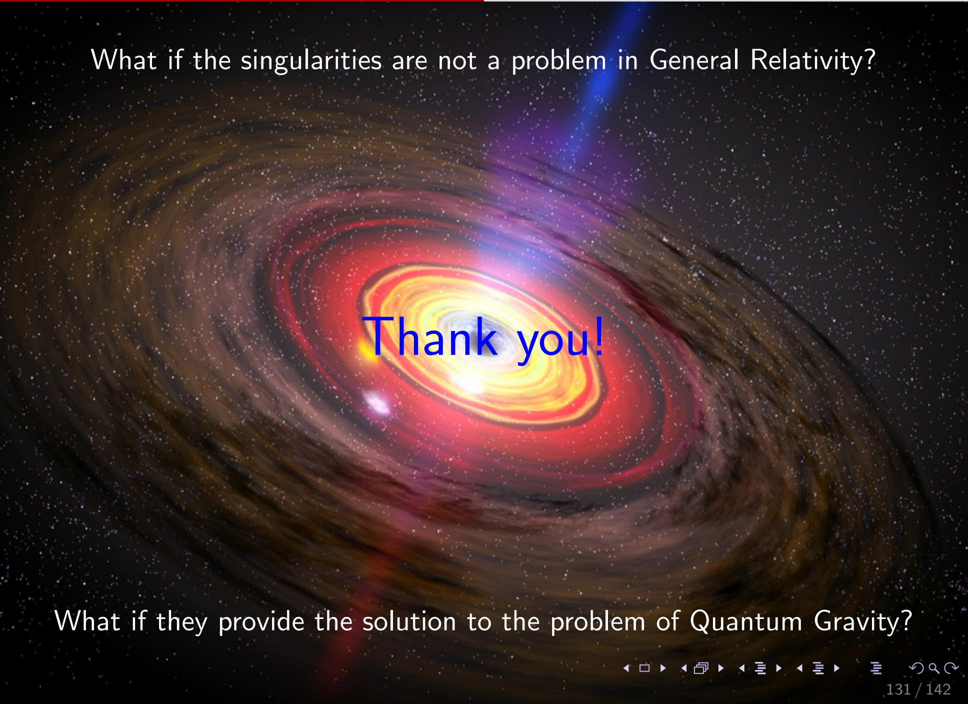
Quantum gravity from dimensional reduction at singularities[23]

To make GR renormalizable, some authors proposed various modifications, entailing apparently distinct kinds of dimensional reduction.

We have seen that some of these are obtained naturally, without modifying GR, from the properties of singularities.

[23](Stoica, 2012d)

What if the singularities are not a problem in General Relativity?



Thank you!

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